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ON THE ALMOST SURE MINIMAL GROWTH RATE OF PARTIAL SUM MAXIMA

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Let $S_n = X_1 + \cdots + X_n$ be partial sums of independent identically distributed random variables and let a_n be an increasing sequence of positive constants tending to ∞. This paper concerns the almost sure lower limit of $\max_{1 \le j \le n} S_j/a_n$. We prove that the lower limit is either 0 or ∞ under mild conditions and give integral tests to determine which is the case. Let $\tau = \inf\{n \ge 1: S_n > 0\}$ and $\tau_- = \inf\{n \ge 1: S_n \le 0\}$. Several inequalities are given that determine up to scale constants various quantities involving truncated moments of the ladder variables S_τ and τ under three different conditions: $ES_\tau < \infty$, $E|S_{\tau_-}| < \infty$ and X symmetric. Moments of ladder variables are also discussed.

1. Introduction. Let $X, X_1, X_2, \ldots$ denote a sequence of independent indentically distributed (iid) nondegenerate random variables. For the random walk generated by X and its partial sum maxima, we use the notation

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \qquad S_n^* = \max_{0 \le j \le n} S_j, \qquad n \ge 1.$$

By the Hewitt–Savage zero-one law, for any normalizing constants $a_n > 0$ there exists a constant $0 \le v \le \infty$ such that

$$\liminf_{n \to \infty} S_n^*/a_n = v \quad \text{a.s.} \tag{1.1}$$

The purpose of this paper is to study the almost sure lower limit v when a_n is nondecreasing and tends to ∞. This is the only nontrivial case, as $\liminf_n S_n^*/a_n^* = \liminf_n S_n^*/a_n$, where $a_n = \max_{1 \le i \le n} a_i$. Among other things, we show that the value of v is always either 0 or ∞ under mild conditions, and give integral tests to determine which is the case. Under various conditions on X, we also obtain inequalities that bound moments and truncated moments of ladder variables associated with the random walk $\{S_n\}$.

Our problem has its origin in the desire on the part of many authors to refine, extend and achieve a deeper understanding of the strong law of large numbers (SLLN) and the law of the iterated logarithm. The specific question of concern

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here was introduced in 1965 by Hirsch, who considered independent mean zero variables. In the iid case, his result becomes

$$(1.2)\quad v=\infty \ \text{ if } \ \sum a_n n^{-3/2}<\infty \quad \text{and} \quad v=0 \ \text{ if } \ \sum a_n n^{-3/2}=\infty,$$

provided that $EX=0$ and $E|X|^3<\infty$. In this paper, we study the almost sure lower limit v under the general condition

$$(1.3)\qquad \limsup_{n\to\infty} S_n=\infty \quad \text{and} \quad \liminf_{n\to\infty} S_n=-\infty \quad \text{a.s.}$$

We shall use the notation $a\vee b=\max(a,b)$, $a\wedge b=\min(a,b)$, $x^+=x\vee 0$ and $x^-=(-x)\vee 0$. We shall use the sign $\sim$ to indicate that the ratio of two sides tends to a finite positive constant.

Define

$$(1.4)\qquad \begin{gathered}\tau=\inf\{n\ge 1\colon S_n>0\},\qquad \tau_0=\inf\{n\ge 1\colon S_n\ge 0\},\\ \tau_-=\inf\{n\ge 1\colon S_n\le 0\}.\end{gathered}$$

Let (Y_k,τ_k), $k\ge 1$, be iid copies of (S_τ,τ),

$$(1.5)\qquad Y_k=S_{T_k}-S_{T_{k-1}},\qquad \tau_k=T_k-T_{k-1},$$

where

$$(1.6)\qquad T_k=\inf\{n>T_{k-1}\colon S_n>S_{T_{k-1}}\},\qquad T_0=0.$$

Because $S_n\le S_{T_k}$ for $T_k\le n<T_{k+1}$,

$$(1.7)\quad \liminf_{n\to\infty} S_n^*/a_n=\liminf_{k\to\infty} S_{T_k}/a(T_{k+1}-1)=\left[\limsup_{k\to\infty} a(T_{k+1}-1)/S_{T_k}\right]^{-1},$$

where $a(\cdot)$ is the linear interpolation of $\{a_n\}$. This gives the connection between the value of v and the ladder variables S_τ and τ.

Define

$$(1.8)\qquad J=\int_0^\infty \frac{x\,dP\{a(\tau)\le x\}}{\int_0^x P\{S_\tau>y\}\,dy}=\sum_{n=1}^\infty \frac{a_n P\{\tau=n\}}{E(S_\tau\wedge a_n)}.$$

As our first theorem indicates, the series expression given by J essentially determines the value of v.

THEOREM 1.1. *Suppose* (1.3) *holds. Let* v *be given by* (1.1) *with constants* a_n, $n\ge 1$, *such that* a_n *is increasing and* a_n/n *is decreasing. Then*

$$(1.9)\qquad v=\infty \quad \text{if } J<\infty \quad \text{and} \quad v=0 \quad \text{if } J=\infty.$$

COROLLARY 1.2. *Suppose $EX = 0$ and $EX^2 < \infty$. Then* (1.2) *holds. In particular*,

$$\liminf_{n\to\infty} \frac{S_n^*}{\sqrt{n}(\log n)^\beta} = \begin{cases} \infty, & \text{if } \beta < -1, \\ 0, & \text{if } \beta \ge -1. \end{cases}$$

PROOF. Because $EX = 0$ and $EX^2 < \infty$,

$$\lim_{n\to\infty} \sqrt{n}P\{\tau > n\} = \frac{e^{-c}}{\sqrt{\pi}} \quad \text{and} \quad ES_\tau = e^{-c}\sqrt{EX^2/2},$$

where $c = \sum_{n=1}^{\infty} n^{-1}[P\{S_n > 0\} - 1/2] \in (-\infty, \infty)$ [cf. Feller (1971), pages 415 and 612]. Hence, by the monotonicity of a_n,

$$J < \infty \quad \text{iff } \sum n^{-1/2}(a_n - a_{n-1}) < \infty \quad \text{iff } \sum a_n n^{-3/2} < \infty. \qquad \square$$

Theorem 1.1 is a consequence of Theorem 2.1 in Section 2, which also covers arbitrary nondecreasing a_n under a mild condition on the distribution of τ. See Example 5.2 for a_n which grows arbitrarily rapidly. Though the integral test J may appear somewhat mystifying, it has an intuitive content that can be made fairly clear. Let

$$m(y) = \sup\{m\colon yE(S_\tau \wedge m) \ge m\}, \qquad y \ge 1.$$

The quantity $m(k)$ represents the typical rate at which the random walk $S_{T_k} = Y_1 + \cdots + Y_k$ grows in the sense that (as we show in Lemma 2.3)

$$P\{S_{T_k} < \tfrac{4}{3}m(k)\} \ge \tfrac{1}{4} \quad \text{for } k \ge 2 \quad \text{and} \quad P\{S_{T_k} \le \tfrac{1}{2}m(k)\} \le \sqrt{2/e}.$$

By inspection of (1.8) and the definition of $m(\cdot)$, we see that $J = Em^{-1}(a(\tau))$. Invoking the Borel–Cantelli lemma and standard results on computation of expectations, we have

$$J < \infty \quad \text{iff } P\{a(\tau_{k+1}) > cm(k) \text{ i.o.}\} = 0 \tag{1.10}$$

for any (and all) $c > 0$ due to the monotonicity of $m(k)/k$. Now, because $m(k)$ provides the order of magnitude of a suitable percentile of S_{T_k}, it is not surprising that for all $c > 0$,

$$P\{a(\tau_{k+1}) > cS_{T_k} \text{ i.o.}\} = P\{a(\tau_{k+1}) > m(k) \text{ i.o.}\}$$

[this is parts (i) and (ii) of Theorem 2.1] and hence

$$J < \infty \quad \text{iff } P\{a(\tau_{k+1}) > cS_{T_k} \text{ i.o.}\} = 0$$

for any (and all) $c > 0$. Because $\tau_{k+1} < T_{k+1} - 1$, $J = \infty$ implies $P\{a(T_{k+1} - 1) > cS_{T_k} \text{ i.o.}\} = 1$ for all $c > 0$. Due to the results of Feller (1946) in the iid infinite mean case, we expect that when τ_{k+1} is large, $T_{k+1}-1$ will be of no larger order. Hence, $J < \infty$ should imply [in view of (1.7)] that

$$P\{a(T_{k+1}-1) > cS_{T_k} \text{ i.o.}\} = P\{a(\tau_{k+1}) > S_{T_k} \text{ i.o.}\}$$

for any (and all) $c > 0$ even when the latter probability is zero, and this is the essential content of Theorem 1.1.

The integral test J has a form analogous to the integral tests of Erickson (1973) and Chow and Zhang (1986). Hence, their probabilistic content is essentially the same. For example, if $E|X| = \infty$, Erickson showed that for any (and all) finite c,

$$\limsup_{n\to\infty} \frac{X_{n+1}}{X_1^- + \cdots + X_n^-} > c \quad \text{a.s. iff} \int_0^\infty \frac{x\,dP\{X \le x\}}{\int_0^x P\{-X > y\}\,dy} = \infty.$$

Put $m_-(y) = \sup\{m\colon yE(X^- \wedge m) \ge m\}$. Then $m_-(n)$ represents the typical growth rate of $\sum_{j=1}^n X_j^-$ and Erickson's result says

$$P\left\{X_{n+1} > c\sum_{j=1}^n X_j^- \text{ i.o.}\right\} = P\{X_{n+1} > m_-(n) \text{ i.o.}\}.$$

To determine whether J is finite or not, we need to obtain computable information concerning the marginal distributions of the ladder variables S_τ and τ. In most cases, it suffices to know the order of the truncated moments $E(S_\tau \wedge x)$ and $E(\tau \wedge n)$. We shall find these orders and derive equivalent integral tests in terms of the distribution of X itself for three families of distributions: (i) when $ES_\tau < \infty$, (ii) when $E|S_{\tau_-}| < \infty$ and (iii) when X is symmetric.

In Section 3 we assume $ES_\tau < \infty$. Because $E(S_\tau \wedge x) \to ES_\tau < \infty$, $J < \infty$ iff $Ea(\tau) < \infty$. It turns out that the inequalities

$$ES_n^+/2n \le ES_{\tau\wedge n}^+/E(\tau \wedge n) \le 4ES_n^+/n \tag{1.11}$$

hold for all distributions with $EX = 0$. This gives the order of $E(\tau \wedge n)$ for the case $ES_\tau < \infty$, because $ES_{\tau\wedge n}^+ = O(1)$ and the order of ES_n^+ was obtained by Klass (1980).

In Section 4 we assume $E|S_{\tau_-}| < \infty$. The order of $E(\tau_- \wedge n)$ is obtained by (1.11) with a change of the sign, and the orders of $E(S_\tau \wedge x)$ and $E(\tau \wedge n)$ are obtained based on the order of $E(\tau_- \wedge n)$ and the duality inequalities

$$n \le E(\tau \wedge n)E(\tau_- \wedge n) \le 2n \tag{1.12}$$

and

$$\frac{1}{2}E(S_\tau \wedge x) \le \int_0^\infty \frac{y(y\wedge x)}{E(|S_{\tau_-}| \wedge y)}\,dP\{X \le y\} \le 2E(S_\tau \wedge x). \tag{1.13}$$

In Section 5 we consider symmetric random variables and obtain

$$\tfrac{1}{2}\sqrt{E(X^2 \wedge x^2)} \le \sqrt{P\{S_{\tau_0} > 0\}}E(S_\tau \wedge x) \le \left(\tfrac{3}{2}\right)^{1/2}\sqrt{E(X^2 \wedge x^2)}. \tag{1.14}$$

The order of $E(\tau \wedge n)$ is the same as that of $E(\tau_- \wedge n)$ in this case and is given by (1.12).

In Section 6 we utilize inequalities (1.11)–(1.14) to investigate moments of ladder variables. Section 7 contains additional discussion.

2. An integral test based on ladder variables. Let $Y_k, \tau_k, T_k, k \geq 1$, be given by (1.5) and (1.6) and let

$$u = \limsup_{k \to \infty} a(\tau_{k+1})/S_{T_k} \quad \text{a.s.} \tag{2.1}$$

THEOREM 2.1. *Suppose* (1.3) *holds. Let v and u be given by* (1.1) *and* (2.1), *respectively, with an increasing sequence of constants* $0 < a_n \to \infty$.

(i) *If $J = \infty$, then $u = \infty$.*
(ii) *If $J < \infty$, then $u = 0$.*
(iii) *If $J = \infty$, then $v = 0$.*
(iv) *If $J < \infty$ and a_n/n is decreasing, then $v = \infty$.*
(v) *If $J < \infty$ and*

$$\limsup_{n \to \infty} E(\tau \wedge n)/\big(nP\{\tau \geq n\}\big) < \infty, \tag{2.2}$$

then $v = \infty$.

REMARKS. (1) By (iii)–(v) of Theorem 2.1, (1.9) holds if either a_n/n is decreasing or (2.2) is satisfied.

(2) If $E(\tau \wedge n) \sim E(\tau_- \wedge n)$ (e.g., symmetric X), then (2.2) holds [cf. (4.1), (3.9), (5.9) and (5.10)].

LEMMA 2.2. *Let $Z, Z_1, Z_2, \dots,$ be iid nonnegative random variables with $P\{Z > 0\} > 0$. Define $U(t) = 1 + \sum_{n=1}^{\infty} P\{Z_1 + \cdots + Z_n \leq t\}, t \geq 0$. Then, for all $t \geq 0$,*

$$t \leq U(t) \int_0^t P\{Z > x\}\, dx \leq 2t,$$

and for all $a > 0$ and $t \geq 0$,

$$\min\big(1, a/2\big) U(t) \leq U(at) \leq \max(1, 2a) U(t).$$

A proof of Lemma 2.2 can be found in Erickson (1973). To make the paper self-contained, here is a simple proof.

PROOF. Let

$$\tau(t) = \inf\{n \geq 1\colon\ Z_1 + \cdots + Z_n > t\}$$

and

$$\tau_-(t) = \inf\big\{n \geq 1\colon\ (Z_1 \wedge t) + \cdots + (Z_n \wedge t) \geq t\big\}.$$

Then $E\tau_-(t) \le E\tau(t) = U(t) \le E\tau_-(t+\varepsilon)$ for all $\varepsilon > 0$, so that the first statement follows from the Wald equation

$$t \le E\tau_-(t)E(Z \wedge t) = E\{Z_1 + \cdots + Z_{\tau_-(t)}\} \le 2t$$

and the continuity of $E(Z \wedge t)$ in t. The second statement is an immediate consequence of the first one. □

LEMMA 2.3. *Let $Y, Y_1, \ldots, Y_k$ be iid nonnegative random variables. Define $m_y = \sup\{m\colon yEY \wedge m \ge m\}$. Set $S = Y_1 + \cdots + Y_k$. Then,*

$$P\{S \le cm_k\} \ge \min\left(\left(1 - \frac{1}{k}\right)^k, 1 - \frac{1}{c}\right) \qquad \forall\, c \ge 1 \textit{ and } k \ge 1,$$

and

$$P\{S \le \delta m_k\} \le \exp[-\delta \log \delta + \delta - 1] \le \exp\left[-\frac{(1-\delta)^2}{2}\right] \qquad \forall\, 0 \le \delta \le 1.$$

PROOF. Set $Y_i' = Y_i \wedge m_k$ and $S' = Y_1' + \cdots + Y_k'$. Let $p = P\{Y > m_k\}$ and let P^* be the conditional probability given $S = S'$. Because $m_k = kE(Y \wedge m_k)$, $p \le k^{-1}$, and $E^*Y_i = (1-p)^{-1}(m_k/k - m_k p)$, so that

$$\begin{aligned}
P\{S \le cm_k\} &\ge (1-p)^k P^*\{S' \le cm_k\} \\
&\ge (1-p)^k\left(1 - \frac{E^*S'}{cm_k}\right) \\
&= (1-p)^k\left(1 - \frac{1-kp}{c(1-p)}\right) \\
&= (1-p)^{k-1}\frac{c - 1 + (k-c)p}{c}.
\end{aligned}$$

Because the logarithm of the right-hand side is concave in p, the minimum is attained at $p = 0$ or $p = 1/k$, which gives the first inequality. For the second inequality, we have

$$P\{S \le \delta m_k\} \le P\{S' \le \delta m_k\} \le \exp(\delta t) E \exp(-tS'/m_k).$$

Due to the convexity of $\exp(-ty/m_k)$ in y, the maximum of $E\exp(-tY'/m_k)$ (subject to $0 \le Y' \le m_k$ and $EY' = m_k/k$) is attained by the distribution with $P\{Y' = m_k\} = 1/k$, so that

$$P\{S \le \delta m_k\} \le e^{\delta t}\left(1 - \frac{1}{k} + \frac{1}{k}e^{-t}\right)^k \le \exp[\delta t + e^{-t} - 1].$$

The proof is completed by setting $t = -\log \delta$ and taking the Taylor expansion at $\delta = 1$. □

The following technique of integrating by parts is used repeatedly in the rest of the paper: for any nonnegative right-continuous functions $h(\cdot)$ and $G(\cdot)$ on $(0,\infty)$ such that $h(\cdot)$ is nondecreasing with $h(0+) = 0$ and $G(\cdot)$ is nonincreasing with $G(\infty-) = 0$, we have

$$-\int_{t>0} h(t-)\,dG(t) = \int_{t>0} G(t)\,dh(t), \tag{2.3}$$

$$\sum_{n=1}^{\infty} h(n)\big[G(n) - G(n+1)\big] = \sum_{n=1}^{\infty} G(n)\big[h(n) - h(n-1)\big]. \tag{2.4}$$

If either $h(\cdot)$ or $G(\cdot)$ is bounded on $(0,\infty)$, then (2.3) is equivalent to the usual formula $\int_0^\infty h(t-)\,dF(t) = \int_0^\infty (1-F(t))\,dh(t)$ for a distribution function F. Otherwise, we can split the integration $\int h\,dG = \int (h_1 + h_2)\,d(G_1 + G_2)$ into four integrations with $h_1(t) = h(t)\wedge h(1)$, $h_2(t) = (h(t) - h(1))^+$, $G_1(t) = G(t)\wedge G(1)$ and $G_2(t) = (G(t) - G(1))^+$, where $\int h_2\,dG_2 = \int G_2\,dh_2 = 0$.

PROOF OF THEOREM 2.1. (i) For $M > 0$, let $B_k = \{a(\tau_{k+1}) > Mcm(k)\}$ and $A_k = \{S_{T_k} \le cm(k)\}$, where by Lemma 2.3 the constant c can be chosen such that $P\{A_k\} \ge \varepsilon$, $\forall\, k$, for some $\varepsilon > 0$. Then A_k is independent of $B_k, B_{k+1}, \dots$, and by (1.10), $\{B_k \text{ i.o.}\} = 1$. It follows from Lemma 3.2 of Klass (1976) that

$$P\big\{a(\tau_{k+1}) > MS_{T_k} \text{ i.o.}\big\} \ge \varepsilon > 0.$$

The desired conclusion follows from the Hewitt–Savage zero-one law.

(ii) Because $S_{T_k}/k \to ES_\tau > 0$, we only need to consider the case $Ea(\tau) = \infty$. It follows from Theorem 3 of Chow and Zhang (1986) that

$$\sum_{j=0}^{k} a(\tau_{2j+1}) \Big/ \sum_{j=1}^{k} Y_{2j} \to 0 \quad \text{a.s.},$$
$$\sum_{j=1}^{k+1} a(\tau_{2j}) \Big/ \sum_{j=0}^{k} Y_{2j+1} \to 0 \quad \text{a.s.}$$

so that

$$\sum_{j=1}^{k+1} a(\tau_j) \Big/ \sum_{j=1}^{k} Y_j = \sum_{j=1}^{k+1} a(\tau_j)/S_{T_k} \to 0 \quad \text{a.s.} \tag{2.5}$$

(iii) Immediate consequence of (i) and (1.7), because $T_{k+1} - 1 \ge \tau_{k+1}$, $k \ge 1$.

(iv) Because a_n/n is decreasing, $a(T_{k+1}) \le \sum_{j=1}^{k+1} a(\tau_j)$. If $Ea(\tau) = \infty$, then $v = \infty$ by (1.7) and (2.5). Set $b(x) = a^{-1}(\varepsilon x)$. If $Ea(\tau) < \infty$, then $b(x)/x$ is nondecreasing in x and $Eb^{-1}(\tau) = \varepsilon^{-1}Ea(\tau) < \infty$. Because $E\tau = \infty$, it follows from the SLLN of Feller (1946) that

$$P\big\{a(T_{k+1}) > \varepsilon k, \text{i.o.}\big\} = P\left\{\sum_{1}^{k+1} \tau_j > b(k), \text{i.o.}\right\} = 0.$$

The proof is completed by (1.7), because $S_{T_k}/k \to ES_\tau > 0$.

(v) For $\varepsilon > 0$, define

$$I_\varepsilon = \sum_{k=1}^{\infty} E\left[\frac{\tau_{k+1}}{a^{-1}(\varepsilon S_{T_k})} \wedge 1\right].$$

If $I_\varepsilon < \infty$, then $\sum_{k=1}^{\infty}\tau_{k+1}/a^{-1}(\varepsilon S_{T_k}) < \infty$ a.s. so that $T_{k+1}/a^{-1}(\varepsilon S_{T_k}) \to 0$ a.s. by the Kronecker lemma, which implies $P\{a(T_{k+1}) \ge \varepsilon S_{T_k}\text{ i.o.}\} = 0$. Thus, we only need to show $I_\varepsilon < \infty$ for all $0 < \varepsilon < 1$. Now,

$$\begin{aligned}
\frac{\varepsilon}{2} I_\varepsilon &= \frac{\varepsilon}{2}\sum_{k=1}^{\infty}\int_0^1 P\left\{\tau_{k+1} > ta^{-1}(\varepsilon S_{T_k})\right\}dt\\
&= \frac{\varepsilon}{2}\int_0^1\int_0^\infty \sum_{k=1}^{\infty} P\left\{\frac{x}{\varepsilon} > S_{T_k}\right\} dP\left\{a\left(\frac{\tau}{t}\right) \le x\right\} dt\\
&\le \int_0^1\int_0^\infty \frac{x}{E(S_\tau \wedge x)} dP\left\{a\left(\frac{\tau}{t}\right) \le x\right\} dt \quad \text{(by Lemma 2.2)}\\
&\le 1 + \int_0^1\int_0^\infty P\left\{a\left(\frac{\tau}{t}\right) > x\right\} d\frac{x}{E(S_\tau \wedge x)}\, dt \quad [\text{by (2.3)}]\\
&= 1 + \int_0^\infty\int_0^1 P\{\tau > ta^{-1}(x)\}\, dt\, d\frac{x}{E(S_\tau \wedge x)}\\
&= 1 + \int_0^\infty \frac{E(\tau \wedge a^{-1}(x))}{a^{-1}(x)}\, d\frac{x}{E(S_\tau \wedge x)}\\
&= 1 + O(1)\int_0^\infty P\{\tau > a^{-1}(x)\}\, d\frac{x}{E(S_\tau \wedge x)} \quad [\text{by (2.2)}]\\
&= 1 + O(1)\int_0^\infty \frac{x}{E(S_\tau \wedge x)}\, dP\{a(\tau) \le x\} \quad [\text{by (2.3)}].
\end{aligned}$$

Hence, $J < \infty$ implies $I_\varepsilon < \infty$ for all $0 < \varepsilon < 1$ and the proof is complete. □

3. The case $ES_\tau < \infty$. Unless otherwise stated, we shall assume $ES_\tau < \infty$ in this section. Because (1.3) is always assumed and $EX^+ < \infty$, we also have $EX = 0$. As mentioned earlier in the introduction, our conditions imply that $J < \infty$ iff $Ea(\tau) < \infty$, and that the order of $E(\tau \wedge n)$ is related to the order of ES_n^+.

For each $y > 0$ let $K(y)$ be the unique positive real number satisfying

$$yE\left[\left(\frac{X}{K(y)}\right)^2 \wedge \left(\frac{|X|}{K(y)}\right)\right] = 1. \tag{3.1}$$

Then $K(y)/\sqrt{y}$ is increasing and $K(y)/y$ is decreasing. It follows from Klass (1980) that

$$K(n)E|Z_1 - Z_2|\left(1 + e^{-n}\left(\frac{n^n}{n!} - 1\right)\right)^{-1} \le 2K(n),$$

where Z_1 and Z_2 are iid Poisson random variables with common mean 1/2. The quantity $E|Z_1 - Z_2| = 0.673^+$, and the sequence $(1 + e^{-n}(n^n/n! - 1))^{-1}$ is minimized at $n = 4$ with minimum 0.849^+. Taking a convenient value, we have

$$K(n)/2 \leq E|S_n| \leq 2K(n). \tag{3.2}$$

The following theorem is a consequence of Theorem 3.4 presented later in this section.

THEOREM 3.1. *Suppose $ES_\tau < \infty$. Let a_n, $n \geq 1$, be positive constants such that a_n/n is decreasing and a_n/n^ε is increasing for some $0 < \varepsilon < 1$. Let J be given by* (1.8). *Then*

$$J < \infty \quad \text{iff} \quad \sum_{n=1}^{\infty} \frac{n}{K(n)} \left(\frac{a_n}{n} - \frac{a_{n+1}}{n+1} \right) < \infty.$$

REMARK. Chow (1986) proved that $ES_\tau < \infty$ if and only if

$$\int_0^\infty \frac{x^2}{\int_0^\infty y(y \wedge x)\, dP\{-X \leq y\}}\, dP\{X \leq x\} < \infty.$$

In particular, $ES_\tau < \infty$ if $P\{-X > y\} \sim y^{-p}(\log y)^\alpha$ and $P\{X > y\} \sim y^{-p}(\log y)^{\alpha'}$ with $\alpha' < \alpha - 1$ and $EX = 0$.

EXAMPLE 3.2. Suppose $P\{-X > y\} \sim y^{-p}(\log y)^\alpha$ and either $E(X^+)^2 < \infty$ or $P\{X > y\} \sim y^{-p}(\log y)^{\alpha'}$ with $\alpha' < \alpha - 1$, where p and α are constants such that $1 \leq p \leq 2, \alpha < -1$ if $p = 1$ and $\alpha \geq -1$ if $p = 2$. Then, $ES_\tau < \infty, K(y) \sim [y(\log y)^\alpha]^{1/p}$ for $1 < p < 2$ and $K(y) \sim [y(\log y)^{\alpha+1}]^{1/p}$ for $p = 1$ or 2. It follows from Theorem 3.1 that

$$\liminf_{n \to \infty} \frac{S_n^*}{n^{1/p}(\log n)^\beta} = \begin{cases} \infty, & \beta < \beta^*, \\ 0, & \beta \geq \beta^*, \end{cases}$$

where

$$\beta^* = \begin{cases} \alpha + 1, & p = 1, \\ (\alpha/p) - 1, & 1 < p < 2, \\ (\alpha - 1)/2, & p = 2. \end{cases}$$

Lemma 3.3 below enables us to approximate $E(\tau \wedge n)$ in conjunction with (3.2).

LEMMA 3.3. *Let τ be given by* (1.4). *Suppose $0 \leq EX \leq \infty$. Then*

$$\frac{ES_n^*}{2n} \leq \frac{ES_{\tau \wedge n}^+}{E(\tau \wedge n)} \leq \frac{ES_{2n}^*}{n} \leq \frac{2ES_n^*}{n}, \qquad n \geq 1. \tag{3.3}$$

REMARKS. (i) We shall also use the analogous formula for $E(\tau_- \wedge n)$ and $ES^-_{\tau_- \wedge n}$.

(ii) By Klass (1989),

$$ES_n^+ \le ES_n^* \le 2ES_n^+, \qquad n \ge 1. \tag{3.4}$$

PROOF. Define

$$Y_{k,n} = S_{T_{k-1}+(\tau_k \wedge n)} - S_{T_{k-1}}, \qquad k \ge 1,$$

and

$$\sigma_n = \inf\{k \ge 1\colon T_k = \tau_1 + \cdots + \tau_k \ge n\}.$$

Then $n \le \sum_{j=1}^{\sigma_n}(\tau_j \wedge n) \le 2n$, so that by Wald's equation,

$$n \le E\sigma_n E(\tau \wedge n) \le 2n. \tag{3.5}$$

Moreover,

$$\max_{0 \le j \le n} S_j \le \sum_{k=1}^{\sigma_n} Y_{k,n}^+ \le \max_{0 \le j \le 2n} S_j.$$

Hence,

$$ES_n^* \le E\sigma_n EY_{1,n}^+ \le ES_{2n}^* \le 2ES_n^*. \tag{3.6}$$

Because $EY_{1,n}^+ = ES^+_{\tau \wedge n}$, we have (3.3) by inserting (3.5) into (3.6). □

Instead of proving Theorem 3.1 directly, we have the following stronger theorem.

THEOREM 3.4. *Suppose $ES_\tau < \infty$. Let $a_n, n \ge 1$, be positive constants such that a_n is increasing and a_n/n is decreasing. Let J be given by* (1.8).

(i) *If $\sum_{n=1}^\infty a_n/(nK(n)) < \infty$, then $J < \infty$.*

(ii) *If there exists a constant $M < \infty$ such that $\sum_{j=n}^\infty a_j/j^2 \le Ma_n/n$ for all $n \ge 1$, then $J < \infty$ iff $\sum_{n=1}^\infty a_n/(nK(n)) < \infty$.*

(iii) *If there exists a constant $M < \infty$ such that $\sum_{j=1}^n a_j/j \le Ma_n$ for all $n \ge 1$, then*

$$J < \infty \quad \textit{iff} \quad \sum_{n=2}^\infty \left(\frac{n}{K(n)} - \frac{n-1}{K(n-1)}\right)\frac{a_n}{n} < \infty.$$

(iv) *If $2a_n \ge a_{n+1} + a_{n-1}, n \ge 2$, then*

$$J < \infty \quad \textit{iff} \quad \sum_{n=2}^\infty \left(\frac{n}{K(n)} - \frac{n-1}{K(n-1)}\right)(a_n - a_{n-1}) < \infty.$$

REMARKS. (i) If a_n/n^ε is increasing for some $0 < \varepsilon < 1$, then

$$\sum_{j=1}^{n} a_j/j \le \frac{a_n}{n^\varepsilon}\sum_{j=1}^{n} j^{\varepsilon-1} \le \frac{a_n}{n^\varepsilon}\int_0^n x^{\varepsilon-1}\,dx = \frac{a_n}{\varepsilon},$$

so that Theorem 3.4(iii) and (2.4) imply Theorem 3.1. Likewise, the condition for Theorem 3.4(ii) holds if $a_n/n^{1-\varepsilon}$ is decreasing for some $\varepsilon > 0$.

(ii) In order to use the bounds of $E(\tau \wedge n)$ in Lemma 3.3, we have to integrate by parts twice to translate $P\{\tau = n\}$ in (1.8) into $E(\tau \wedge n)$. This caused us to consider several different conditions on a_n.

We shall list a few facts that are useful here and in later sections. Let Z be a nonnegative random variable and b_n, k_n, $n \ge 1$, be positive constants such that b_n/n is decreasing, $k_n/\sqrt{n}$ is increasing and $c_1 k_n \le EZ \wedge n \le c_2 k_n$ for some constants $0 < c_1 < c_2 < \infty$. Then the following inequalities hold:

$$b_n - b_{n-1} \le b_n - (n-1)b_n/n = b_n/n, \tag{3.7}$$

$$k_n - k_{n-1} \ge k_n - \sqrt{n-1}\,k_n/\sqrt{n} \ge k_n/(2n) \tag{3.8}$$

and

$$(4c_2)^{-1}c_1^2 k_n \le nP\{Z \ge n\} \le c_2 k_n. \tag{3.9}$$

The first inequality of (3.9) is a consequence of

$$(m-1)nP\{Z \ge n\} \ge E\big(Z \wedge (mn)\big) - E(Z \wedge n) \ge c_1 k_{mn} - c_2 k_n \ge (c_1\sqrt{m} - c_2)k_n,$$

as $(c_1\sqrt{m} - c_2)/(m-1) \ge (4c_2)^{-1}c_1^2$ with m being the integer part of $4(c_2/c_1)^2 + 1$.

PROOF OF THEOREM 3.4. Because $ES_\tau < \infty, E(S_\tau \wedge a_n) \to ES_\tau < \infty$, so that by (1.8), $J < \infty$ iff $Ea_\tau < \infty$. Because a_n is increasing and a_n/n is decreasing, by (2.4),

$$Ea_\tau = \sum_{n=1}^{\infty} P\{\tau \ge n\}(a_n - a_{n-1}). \tag{3.10}$$

It follows from Lemma 3.3, (3.2) and (3.4) that there exist constants $0 < c_1 < c_2 < \infty$ (dependent on the distribution of X but not on n) such that

$$c_1 n/K(n) \le E(\tau \wedge n) \le c_2 n/K(n). \tag{3.11}$$

(i) By (3.7) (with $a_n = b_n$), (3.10) and (3.11),

$$Ea_\tau \le \sum_{n=1}^{\infty} P\{\tau \ge n\}a_n/n \le \sum_{n=1}^{\infty} E(\tau \wedge n)a_n/n^2 \le c_2 \sum_{n=1}^{\infty} a_n/\big(nK(n)\big).$$

(ii) Because $\Sigma_{j=n}^{\infty} a_n/n^2 < \infty$, $a_n/n \to 0$. Because

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n} jP\{\tau = j\}\right)&(a_n - a_{n-1})/n \\ &= \sum_{j=1}^{\infty} jP\{\tau = j\}\sum_{n=j}^{\infty}(a_n - a_{n-1})/n \\ &\le \sum_{j=1}^{\infty} jP\{\tau = j\}\sum_{n=j}^{\infty} a_n/n^2 \quad \text{[by (3.7)]} \\ &\le MEa_\tau,\end{aligned}$$

by (3.10), $Ea_\tau \le \Sigma_{n=1}^{\infty} E(\tau \wedge n)(a_n - a_{n-1})/n \le (M+1)Ea_\tau$, so that by (3.11) and (2.4),

$$\begin{aligned}Ea_\tau < \infty \quad &\text{iff } \sum_{n=1}^{\infty}(a_n - a_{n-1})/K(n) < \infty, \\ &\text{iff } \sum_{n=1}^{\infty}\big(1/K(n) - 1/K(n+1)\big)a_n < \infty.\end{aligned}$$

The conclusion follows from (3.7) and (3.8) with $b_n = k_n = K(n)$.

(iii) Because $a_n \le \Sigma_{j=1}^{n} a_j/j \le Ma_n$, by (2.4) we have

$$\begin{aligned}Ea_\tau < \infty \quad &\text{iff } \sum_{n=1}^{\infty} P\{\tau = n\}\sum_{j=1}^{n} a_j/j < \infty, \\ &\text{iff } \sum_{n=1}^{\infty}\frac{a_n}{n}P\{\tau \ge n\} < \infty, \\ &\text{iff } \sum_{n=1}^{\infty} E(\tau \wedge n)\left(\frac{a_n}{n} - \frac{a_{n+1}}{n+1}\right) < \infty.\end{aligned}$$

The proof is completed by (3.11) and the monotonicity of $n/K(n)$.

(iv) Because a_n is concave, we have by (3.10), (2.4) and (3.11),

$$\begin{aligned}Ea_\tau < \infty \quad &\text{iff } \sum_{n=2}^{\infty} E(\tau \wedge n)(2a_n - a_{n+1} - a_{n-1}) < \infty, \\ &\text{iff } \sum_{n=2}^{\infty}\frac{n}{K(n)}(2a_n - a_{n+1} - a_{n-1}) < \infty.\end{aligned}$$ □

4. The case $E|S_{\tau_-}| < \infty$. We shall assume $E|S_{\tau_-}| < \infty$ in this section. Again, this and (1.3) give us $EX = 0$. Proceeding in a similar manner as in Section 3, we obtain bounds for $E(\tau_- \wedge n)$. A duality lemma below connects the truncated moments of ascending and descending ladder variables, so that we can find the order of $E(S_\tau \wedge x)$ and $E(\tau \wedge n)$.

THEOREM 4.1. *Let J be given by* (1.8). *Suppose $E|S_{\tau_-}| < \infty$ and $a_n > 0$ is increasing in n. Then*

$$J < \infty \quad \text{iff} \quad \sum_{n=1}^{\infty} \frac{a_n}{E\big(X^+(X^+ \wedge a_n)\big)} \left(\frac{K(n)}{n} - \frac{K(n+1)}{n+1} \right) < \infty.$$

EXAMPLE 4.2. Suppose $P\{X > y\} \sim y^{-p}(\log y)^{\alpha}$ and either $E(X^-)^2 < \infty$ or $P\{-X > y\} \sim y^{-p}(\log y)^{\alpha'}$ with $\alpha' < \alpha - 1$, where $1 \le p \le 2$, $\alpha < -1$ if $p = 1$, and $\alpha \ge -1$ if $p = 2$. Then the function $K(\cdot)$ is as in Example 3.2 so that

$$\liminf_{n \to \infty} \frac{S_n^*}{n^{1/p}(\log n)^{\beta}} = \begin{cases} \infty, & \beta < \beta^*, \\ 0, & \beta \ge \beta^*, \end{cases}$$

where

$$\beta^* = \begin{cases} \alpha/p - 1/(p-1), & 1 < p < 2, \\ (\alpha - 1)/2, & p = 2 \end{cases}$$

and for $p = 1$,

$$\liminf_{n \to \infty} \frac{S_n^*}{\exp\big[(\log n)(\log\log n)^{\beta}\big]} = \begin{cases} \infty, & \beta < -1/|\alpha + 1|, \\ 0, & \beta \ge -1/|\alpha + 1|. \end{cases}$$

It is worthwhile to observe here that the critical normalizing sequence for $p = 1$ is much smaller than those for $1 < p \le 2$.

LEMMA 4.3. *Let τ and τ_- be defined by* (2.1). *Suppose $EX = 0$. Then*

$$n \le E(\tau \wedge n)E(\tau_- \wedge n) \le 2n \tag{4.1}$$

and

$$\frac{1}{2}E(S_\tau \wedge x) \le \int_0^{\infty} \frac{y(y \wedge x)}{E\big(|S_{\tau_-}| \wedge y\big)}\, dP\{X \le y\} \le 2E(S_\tau \wedge x). \tag{4.2}$$

REMARKS. (1) The double inequality (4.1) holds without the assumption that $EX = 0$.

(2) It follows from Lemmas 3.3 and 4.3 and (3.2) that for mean zero random walks,

$$K^2(n)/(64n) \le ES^+_{\tau \wedge n} ES^-_{\tau_- \wedge n} \le 32K^2(n)/n.$$

(3) Chow (1986) proved (4.2) up to a scale constant.

(4) Because both $y/E(|S_{\tau_-}|\wedge y)$ and $E(|S_{\tau_-}|\wedge y)$ are increasing in y, (4.2) implies that for all positive numbers x_1 and x_2,

$$2E(S_\tau\wedge x_1)E\big(|S_{\tau_-}|\wedge x_2\big)\geq\int_0^\infty(y\wedge x_1)(y\wedge x_2)\,dP\{X\leq y\}. \tag{4.3}$$

Consequently, we also have

$$2E(S_\tau\wedge x_1)\big(|S_{\tau_-}|\wedge x_2\big)\geq\int_0^\infty(y\wedge x_1)(y\wedge x_2)\,dP\{-X\leq y\}.$$

PROOF. Let τ_k and T_k be defined by (1.5) and (1.6). It follows from the duality principle of random walks [Feller (1971), page 394] that

$$P\{\tau_->n\}=P\{S_n>S_j,\ 0\leq j\leq n-1\}=\sum_{k=0}^\infty P\{T_k=n\},$$

so that

$$E(\tau_-\wedge n)=\sum_{j=0}^{n-1}P\{\tau_->j\}=\sum_{k=0}^\infty P\{\tau_1+\cdots+\tau_k<n\},$$

which implies (4.1) by Lemma 2.2. For (4.2) we have

$$\begin{aligned}
&P\{|S_{\tau_-}|>x\}\\
&\quad=\sum_{n=1}^\infty P\{S_1>0,\dots,S_{n-1}>0,\ S_n<-x\}\\
&\quad=\int_{-\infty}^{-x}\left[1+\sum_{n=2}^\infty P\{S_1>0,\dots,S_{n-1}>0,\ S_{n-1}<-x-y\}\right]dP\{X\leq y\}\\
&\quad=\int_x^\infty\left[1+\sum_{n=1}^\infty P\{S_n>S_j,\ 0\leq j\leq n-1,\ S_n>0,\ S_n<y-x\}\right]\\
&\qquad\quad\times dP\{-X\leq y\}\\
&\quad=\int_x^\infty\sum_{k=0}^\infty P\{S_{T_k}<y-x\}\,dP\{-X\leq y\}.
\end{aligned}$$

Therefore, for $c>0$

$$E\big(|S_{\tau_-}|\wedge c\big)=\int_0^\infty\int_0^{y\wedge c}\sum_{k=0}^\infty P\{S_{T_k}<y-x\}\,dx\,dP\{-X\leq y\}. \tag{4.4}$$

It follows from Lemma 2.2 that for $0<x<(y\wedge c)$,

$$\sum_{k=0}^\infty P\{S_{T_k}<y-x\}\leq\sum_{k=0}^\infty P\{S_{T_k}<y\}\leq 2y/E(S_\tau\wedge y)$$

and

$$\sum_{k=0}^{\infty} P\{S_{T_k} < y - x\} \geq (y-x)/E\big(S_\tau \wedge (y-x)\big) \geq (y-x)/E(S_\tau \wedge y).$$

Inserting these inequalities into (4.4) and integrating out dx, we have

$$\frac{1}{2} E\big(|S_{\tau_-}| \wedge c\big) \leq \int_0^\infty \frac{y(y \wedge c)}{E(S_\tau \wedge y)} dP\{-X \leq y\} \leq 2E(S_{\tau_-} \wedge c).$$

The proof is complete because the argument still works if we replace X by $-X$ and strict (weak) ladder variables by weak (strict) ladder variables. □

PROOF OF THEOREM 4.1. By (4.2) and the argument leading to (3.11), there exist constants $0 < c_1 < c_2 < \infty$ (dependent on the distribution of X but not on n) such that

$$c_1 E(S_\tau \wedge x) \leq \int_0^\infty y(y \wedge x)\, dP\{X \leq y\} \leq c_2 E(S_\tau \wedge x) \qquad \forall x \geq 0, \tag{4.5}$$

and $c_1 K(n)/n \leq 1/E(\tau_- \wedge n) \leq c_2 K(n)/(2n)$, so that by (4.1), $c_1 K(n) \leq E(\tau \wedge n) \leq c_2 K(n)$. Because $K(y)/\sqrt{y}$ is increasing, by (3.9),

$$(4c_2)^{-1} c_1^2 K(n) \leq nP\{\tau \geq n\} \leq E(\tau \wedge n) \leq c_2 K(n). \tag{4.6}$$

Because both $a_n/E(X^+(X^+ \wedge a_n))$ and $K(n)/n$ are monotone, by (2.4), $J < \infty$ is equivalent to the following statements:

$$\sum_{n=1}^{\infty} P\{\tau = n\} a_n / E\big(X^+(X^+ \wedge a_n)\big) < \infty \quad \big[\text{by } (4.5)\big],$$

$$\sum_{n=2}^{\infty} P\{\tau \geq n\} \Big[a_n/E\big(X^+(X^+ \wedge a_n)\big) - a_{n-1}/E\big(X^+(X^+ \wedge a_{n-1})\big)\Big] < \infty,$$

$$\sum_{n=1}^{\infty} \Big[a_n/E\big(X^+(X^+ \wedge a_n)\big)\Big]\big[K(n)/n - K(n+1)/(n+1)\big] < \infty \quad \big[\text{by } (4.6)\big].$$

□

5. Symmetric case. We shall consider the symmetric case $P\{X > x\} = P\{-X > x\}$ in this section. The order of $E(\tau \wedge n)$ (and therefore $P\{\tau > n\}$) is given by the duality inequality (4.1). The order of $E(S_\tau \wedge x)$ is given by Lemma 5.3.

THEOREM 5.1. *Let J be given by* (1.8) *with an increasing sequence $a_n > 0$. Suppose that X is symmetric* (*with $E|X| < \infty$ or $= \infty$*). *Then* (1.9) *holds* (*whether a_n/n is decreasing or not*) *and*

$$J < \infty \quad \textit{iff} \quad \sum_{n=1}^{\infty} a_n n^{-3/2} \Big/ \sqrt{E\big(X^2 \wedge a_n^2\big)} < \infty. \tag{5.1}$$

EXAMPLE 5.2. Let X be symmetric and $P\{X > y\} \sim y^{-p}(\log y)^{\alpha}$ with $0 \le p \le 2$ and $\alpha \ge -1$ if $p = 2$. Then $E(X^2 \wedge x^2) \sim x^{2-p}(\log x)^{\alpha}$ if $0 \le p < 2$, and $E(X^2 \wedge x^2) \sim (\log x)^{\alpha+1}$ if $p = 2$. It follows from Theorem 5.1 that for $0 < p \le 2$,

$$\liminf_{n\to\infty} \frac{S_n^*}{n^{1/p}(\log n)^{\beta}} = \begin{cases} \infty, & \beta < \beta^*, \\ 0, & \beta \ge \beta^*, \end{cases}$$

where

$$\beta^* = \begin{cases} (\alpha-2)/p, & 0 < p < 2, \\ (\alpha-1)/2, & p = 2 \end{cases}$$

and for $p = 0 > \alpha$,

$$\liminf_{n\to\infty} \frac{S_n^*}{\exp\left[n^{1/|\alpha|}(\log n)^{\beta}\right]} = \begin{cases} \infty, & \beta < 2/\alpha, \\ 0, & \beta \ge 2/\alpha. \end{cases}$$

The following lemma gives an inverse of (4.3) with $x_1 = x_2$ up to a constant scale.

LEMMA 5.3. *Suppoe X is symmetric. Define τ and τ_0 by* (1.4). *Then, for all positive real numbers x,*

$$\frac{1}{2}\sqrt{E(X^2 \wedge x^2)} \le E(S_\tau \wedge x)\sqrt{P\{S_{\tau_0} > 0\}} \le \left(\tfrac{3}{2}\right)^{1/2}\sqrt{E(X^2 \wedge x^2)}. \tag{5.2}$$

REMARK. Clearly, $P\{X > 0\} \le P\{S_{\tau_0} > 0\} \le 1$.

PROOF. We shall first consider the continuous case and then take the limit.

Step 1. Suppose X has a continuous distribution function. Clearly, $P\{\tau_0 = \tau\} = P\{S_{\tau_0} > 0\} = 1$. Because X is symmetric, S_τ has the same distribution as $|S_{\tau-}|$, so that by (4.3) we have

$$\begin{aligned} 2\left[E(S_\tau \wedge x)\right]^2 &= 2E(S_\tau \wedge x)E\left(|S_{\tau-}| \wedge x\right) \\ &\ge E(X^+ \wedge x)^2 = E\left(X^2 \wedge x^2\right)/2. \end{aligned}$$

This gives the first inequality in (5.2).

For the second inequality, let $T = \inf\{n \ge 1: |X_n| > x\}$, $p = P\{|X| \le x\}$ and let P^* be the conditional probability given $|X_n| \le x$, $n \ge 1$. Because

$$P\{\tau < T\} = \sum_{n=1}^{\infty} P\{\tau = n < T\} = \sum_{n=1}^{\infty} p^n P^*\{\tau = n\},$$

by a theorem of Baxter (1985) and the fact that $P^*\{S_n > 0\} = 1/2$,

$$P\{\tau \geq T\} = 1 - E^* p^\tau = \sqrt{1-p}, \tag{5.3}$$

where E^* is the expectation under P^* [cf. Spitzer (1960), page 156]. Also, we have

$$ES_\tau I\{T > \tau\} = \sum_{n=1}^{\infty} E^* S_n I\{\tau = n\} p^n \leq pE^* S_\tau.$$

Because X is a symmetric random variable with a continuous distribution function, by Spitzer's [(1960), page 158] formula, $E^* S_\tau = \sqrt{E^* X^2/2}$. It follows that

$$ES_\tau I\{T > \tau\} \leq p\sqrt{E^* X^2/2} \leq \sqrt{EX^2 I\{|X| \leq x\}/2}.$$

Let $\lambda = x^2 P\{|X| > x\}/E(X^2 \wedge x^2)$. Then $0 \leq \lambda \leq 1$ and by (5.3) and the above inequalities,

$$\begin{aligned} E(S_\tau \wedge x) &\leq xP\{T \leq \tau\} + ES_\tau\{T > \tau\} \\ &\leq x\sqrt{1-p} + \sqrt{EX^2 I\{|X| \leq x\}/2} \\ &= \left[\sqrt{\lambda} + \sqrt{(1-\lambda)/2}\right]\sqrt{E\left(X^2 \wedge x^2\right)}. \end{aligned}$$

Because $\sqrt{\lambda} + \sqrt{(1-\lambda)/2}$ is maximized at $\lambda = 2/3$ with a maximum of $\sqrt{3/2}$, we have the second inequality in (5.2).

Step 2. (Taking the limit.) Let Z_n, $n \geq 1$, be iid standard normal random variables. For $\varepsilon > 0$ define

$$X'_n = X_n + \varepsilon Z_n, \qquad S'_n = X'_1 + \cdots + X'_n, \qquad \tau' = \inf\left\{n \geq 1: S'_n > 0\right\}.$$

Because $E[(X')^2 \wedge x^2] \to E(X^2 \wedge x^2)$ as $\varepsilon \to 0$ and (5.2) holds for $(X', S'_{\tau'})$, it suffices for us to show that

$$E(S'_{\tau'} \wedge x) \to \sqrt{P\{S_{\tau_0} > 0\}}\, E(S_\tau \wedge x) \quad \text{as } \varepsilon \to 0+. \tag{5.4}$$

Let $(Y_{0,k}, \tau_{0,k})$, $k \geq 1$, be iid copies of (S_{τ_0}, τ_0) defined by

$$Y_{0,k} = S_{T_{0,k}} - S_{T_{0,k-1}}, \qquad \tau_{0,k} = T_{0,k} - T_{0,k-1}, \tag{5.5}$$

where $T_{0,k} = \inf\{n > T_{0,k-1}: S_n \geq S_{T_{0,k-1}}\}$ and $T_{0,0} = 0$. It turns out that as $\varepsilon \to 0+$,

$$\tau' \to \tau^* \quad \text{and} \quad S'_{\tau'} \to S_{\tau^*} \quad \text{a.s.} \tag{5.6}$$

with

$$\tau^* = \inf\left\{T_{0,k} \geq 1: S_{T_{0,k}} > 0 \text{ or } Z_1 + \cdots + Z_{T_{0,k}} > 0\right\}. \tag{5.7}$$

Define $\xi_k = Z_{T_{0,k-1}+1} + \cdots + Z_{T_{0,k}}$ and $N = \inf\{k \geq 1: \xi_1 + \cdots + \xi_k > 0\}$. Then

$$(5.8)\quad P\{S_{\tau^*} = 0\} = \sum_{k=1}^{\infty} P\{S_{T_{0,k}} = 0, N = k\} = \sum_{k=1}^{\infty} P^k\{S_{\tau_0} = 0\} P^*\{N = k\},$$

where P^* is the conditional probability given $S_{T_{0,k}} = 0$, $k \geq 1$. Because ξ_k, $k \geq 1$, are iid symmetric continuous random variables under P^*, it follows from Baxter's theorem and the fact $P^*\{\xi_1 + \cdots + \xi_k > 0\} = \frac{1}{2}$ that

$$\sum_{k=1}^{\infty} \theta^k P^*\{N = k\} = E^*\theta^N = 1 - \sqrt{1-\theta}, \qquad 0 \leq \theta \leq 1,$$

so that by (5.8),

$$P\{S_{\tau^*} = 0\} = 1 - \sqrt{P\{S_{\tau_0} > 0\}}.$$

Consequently, by (5.6) and (5.7),

$$E(S'_{\tau'} \wedge x) \to E(S_{\tau^*} \wedge x) = P\{S_{\tau^*} > 0\} E(S_\tau \wedge x) = \sqrt{P\{S_{\tau_0} > 0\}} E(S_\tau \wedge x).$$

This gives (5.4) and the proof is complete. □

PROOF OF THEOREM 5.1. Let $\tau_{0,k}$ and $Y_{0,k}$ be as in (5.5) and $\delta_k = I\{Y_{0,k} = 0\}$. By the definition of τ and τ_0,

$$\tau = \sum_{k=1}^{\infty} \tau_{0,k} \prod_{j=1}^{k-1} \delta_j,$$

so that

$$E(\tau_0 \wedge n) \leq E(\tau \wedge n) \leq \sum_{k=1}^{\infty} P^{k-1}\{\delta = 1\} E(\tau_0 \wedge n) = E(\tau_0 \wedge n)/P\{S_{\tau_0} > 0\}.$$

Because $E(\tau_- \wedge n) = E(\tau_0 \wedge n)$, by Lemma 4.3

$$(5.9)\qquad n \leq E(\tau \wedge n)^2 \leq 2n/P\{S_{\tau_0} > 0\}.$$

By (3.9),

$$(5.10)\qquad \left(4\sqrt{2/P\{S_{\tau_0} > 0\}}\right)^{-1} \leq \sqrt{n} P\{\tau \geq n\} \leq \sqrt{2/P\{S_{\tau_0} > 0\}}.$$

Therefore, (2.2) holds, so that (1.9) holds by Theorem 2.1(iii) and (v).

Because $a_n/E(S_\tau \wedge a_n)$ is increasing, $J < \infty$ is equivalent to the following statements:

$$\sum P\{\tau \geq n\}\left[a_n/E(S_\tau \wedge a_n) - a_{n-1}/E(S_\tau \wedge a_{n-1})\right] < \infty \quad \text{by (2.4);}$$

$$\sum n^{-1/2}\left[a_n/E(S_\tau \wedge a_n) - a_{n-1}/E(S_\tau \wedge a_{n-1})\right] < \infty \quad \text{by (5.10);}$$

$$\sum a_n n^{-3/2} \Big/ \sqrt{E(X^2 \wedge a_n^2)} < \infty \quad \text{by (2.4) and Lemma 5.3.}$$ □

6. Moments of ladder variables. In this section we consider conditions for the finiteness of ES_τ^p and $E\tau^p$ for $0 < p < 1$, which are problems of independent interest.

6.1. *The finiteness of* ES_τ^p. Suppose $EX = 0$. For $p \geq 1$, Chow (1986) proved that $ES_\tau^p < \infty$ iff

$$\int_0^\infty x^{p+1}\left[\int_0^\infty y(y \wedge x)\, dP\{-X \leq y\}\right]^{-1} dP\{X \leq x\} < \infty,$$

which can be written as

$$\int_0^\infty [K_-(y)]^{p-1} y\, dP\{X \leq K_-(y)\} < \infty,$$

where essentially as in (3.1), $K_-(y)$ is defined by

$$yE\left[\left(\frac{X^-}{K_-(y)}\right)^2 \wedge \left(\frac{X^-}{K_-(y)}\right)\right] = 1.$$

For $0 < p < 1$, Chow and Lai (1978) showed that $E(X^+)^{p+1} < \infty$ is a sufficient condition for $ES_\tau^p < \infty$. This sufficient condition was also shown to be necessary by Wolff (1984) under $E(X^-)^2 < \infty$ and by Hogan (1984) under $E|S_{\tau_-}| < \infty$.

THEOREM 6.1. *Let* τ *be given by* (1.4). *Suppose* X *is symmetric* (*with* $E|X| \leq \infty$). *Then, for* $0 < p < 1$,

$$\sqrt{P\{S_{\tau_0} > 0\}}ES_\tau^p \leq \left(\tfrac{3}{2}\right)^{1/2} p|p-1| \int_0^\infty x^{p-2}\sqrt{E(X^2 \wedge x^2)}\, dx$$

and

$$\sqrt{P\{S_{\tau_0} > 0\}}ES_\tau^p \geq \tfrac{1}{2}p|p-1| \int_0^\infty x^{p-2}\sqrt{E(X^2 \wedge x^2)}\, dx.$$

Theorem 6.1 follows immediately from Lemma 5.3 and (2.3). The integration $\int_0^\infty x^{p-2}\sqrt{E(X^2 \wedge x^2)}\,dx$ is finite if $E|X|^{2p+\varepsilon} < \infty$ for some $\varepsilon > 0$, and is infinite if $E|X|^{2p} = \infty$. Thus, our sufficient and necessary condition for $ES_\tau^p < \infty$ is quite different from $E(X^+)^{p+1} < \infty$, when X is symmetric.

EXAMPLE 6.2. Let X be symmetric with $P\{X > y\} \sim y^{-2p}(\log y)^{-2}$, $0 < p < 1$. Then, $E|X|^{2p} < \infty$ and $ES_\tau^p = \infty$.

6.2. *The finiteness of* $E\tau^p$. If X is an integer-valued random variable with $EX = 0$ and $P\{X < -1\} = 0$, then

$$P\{\tau_0 > n \mid S_n\} = S_n^-/n$$

via a generalization of the ballot problem, so that

$$E\tau_0^p = 1 + \sum_{n=1}^{\infty} \left[(n+1)^p - n^p\right] E|S_n|/(2n).$$

Under the general assumption $EX = 0$, Chow (1988) proved that for $1 \le p < 2$,

$$\sum_{n=1}^{\infty} n^{-1-1/p} E|S_n| < \infty$$

iff

$$\int_0^{\infty} \left\{G^{1/p}(t) + \log(1+t)G(t)\right\} dt < \infty,$$

where $G(t) = P\{|X| > t\}$. Based on the ballot problem, he also conjectured that $\sum_{n=1}^{\infty} n^{p-2} E|S_n| < \infty$ is a sufficient and necessary condition for $E\tau^P < \infty$ under the assumption $E(X^-)^2 < \infty$ (private communication). Theorem 6.3(ii) shows that his conjecture is true under the weaker assumption $E|S_{\tau_-}| < \infty$.

THEOREM 6.3. *Let τ be given by* (1.4) *and $K(\cdot)$ by* (3.1). *Suppose $EX = 0$.*

(i) *If $ES_\tau < \infty$, then $E\tau^p < \infty$ for $p < 1/2$ and for $1/2 \le p < 1$,*

$$E\tau^p < \infty \quad \textit{iff} \quad \int_1^{\infty} x^{p-1}\left[K(x)\right]^{-1} dx < \infty.$$

(ii) *If $E|S_{\tau_-}| < \infty$, then $E\sqrt{\tau} = \infty$ and for $0 \le p < 1/2$,*

$$E\tau^p < \infty \quad \textit{iff} \quad \int_1^{\infty} x^{p-2} K(x)\, dx < \infty \quad \textit{iff} \quad \int_1^{\infty} P^{1-p}\{|X| > x\}\, dx < \infty.$$

PROOF. Part (i) follows from Theorem 3.1. Part (ii) follows from (4.6) in the proof of Theorem 4.1 and (3.2):

$$E\tau^p < \infty \quad \text{iff} \quad \sum_{n=1}^{\infty} n^{p-2} K(n) < \infty \quad \text{iff} \quad \sum_{n=1}^{\infty} n^{p-2} E|S_n| < \infty.$$

Because $K(y)/\sqrt{y}$ is increasing, $E\sqrt{\tau} = \infty$. It follows from Chow [(1988), page 180] that

$$\sum_{n=1}^{\infty} \frac{E|S_n|}{n^{\alpha+1}} < \infty \quad \text{iff} \quad \int_0^{\infty} P^{\alpha}\{|X| > x\} dx < \infty, \ \frac{1}{2} < \alpha < 1.$$

Setting $\alpha = 1 - p$ completes the proof. □

7. Remarks. It follows from the methods in Section 3 [i.e., (3.2)–(3.6)] that $ES^+_{\tau \wedge n}/E(\tau \wedge n)$ has the order of $K(n)/n$ as $n \to \infty$. However, $ES^+_{\tau \wedge n}$ is not of the form $E(S_\tau \wedge x_n)$. If we can find a sequence x_n such that $ES^+_{\tau \wedge n}/E(S_\tau \wedge x_n)$ is bounded away from both 0 and ∞, then $E(S_\tau \wedge x_n)$ has the order of $E(\tau \wedge n)K(n)/n$ and the integral test in (1.9) can be determined by the distribution of τ alone. The following proposition shows that $K(n) = x_n$ is such a function under a mild condition on the distribution of X.

PROPOSITION 7.1. *Let τ be given by* (1.4) *and $K(\cdot)$ by* (3.1). *Suppose $EX = 0$. Then, for all $n \geq 1$,*

$$E\big(S_\tau \wedge K(n)\big) \leq 9ES^+_{\tau \wedge n}$$

and

$$ES^+_{\tau \wedge n} \leq (1 + 40M)E\big(S_\tau \wedge K(n)\big),$$

where

$$M = \sup_{n \geq 1} K(n)E\big(X - K(n)\big)^+ / E\big(X^2 \wedge K^2(n)\big).$$

PROOF. By (3.2)–(3.4) we have

$$E(\tau \wedge n)\frac{K(n)}{n} \leq 8ES^+_{\tau \wedge n}, \qquad ES^-_{\tau_- \wedge n} \leq \frac{4K(n)}{n}E(\tau_- \wedge n). \tag{7.1}$$

By the first of the above inequalities,

$$\begin{aligned} E\big[S_\tau \wedge K(n)\big] &\leq ES^+_{\tau \wedge n} + K(n)P\{\tau > n\} \\ &\leq ES^+_{\tau \wedge n} + K(n)E(\tau \wedge n)/n \\ &\leq 9ES^+_{\tau \wedge n}. \end{aligned}$$

To obtain an inequality in the other direction, we have

$$\begin{aligned} ES^+_{\tau \wedge n} &\leq E\big[S_\tau \wedge K(n)\big] + E\sum_{i=1}^{\tau \wedge n} \big(X_i - K(n)\big)^+ \\ &= E\big[S_\tau \wedge K(n)\big] + E(\tau \wedge n)E\big(X - K(n)\big)^+. \end{aligned} \tag{7.2}$$

By (4.3),

$$\begin{aligned} E\big(X - K(n)\big)^+ &\leq \big(M/K(n)\big)E\big(X^2 \wedge K^2(n)\big) \\ &\leq \big(M/K(n)\big)4E\big(S_\tau \wedge K(n)\big)E\big[|S_{\tau_-}| \wedge K(n)\big], \end{aligned} \tag{7.3}$$

while by the second inequality of (7.1),

$$\begin{aligned} E\big[|S_{\tau_-}| \wedge K(n)\big] &\leq ES^-_{\tau_- \wedge n} + K(n)E(\tau_- \wedge n)/n \\ &\leq 5K(n)E(\tau_- \wedge n)/n. \end{aligned} \tag{7.4}$$

Putting (7.2)–(7.4) together, we obtain

$$\begin{aligned} ES^+_{\tau\wedge n} &\le E\big(S_\tau \wedge K(n)\big)\Big[1 + E(\tau\wedge n)\big(4M/K(n)\big)E\big(|S_{\tau_-}| \wedge K(n)\big)\Big] \\ &\le E\big(S_\tau \wedge K(n)\big)\big[1 + 20ME(\tau\wedge n)E(\tau_- \wedge n)/n\big] \\ &\le (1+40M)E\big(S_\tau \wedge K(n)\big) \quad \text{by (4.1).} \end{aligned}$$ □

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