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## A NONLINEAR RENEWAL THEORY

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Let  $T$  be the first time that a perturbed random walk crosses a nonlinear boundary. This paper concerns the approximations of the distribution of the excess over the boundary, the expected stopping time  $ET$  and the variance of the stopping time  $\text{Var}(T)$ . Expansions are obtained by using linear renewal theorems with varying drift.

**1. Introduction.** Let  $X, X_1, X_2, \dots$  be independent identically distributed random variables with  $EX = \mu > 0$  and  $\text{Var}(X) = \sigma^2 > 0$ . Let  $\xi = \{\xi_n, n \geq 1\}$  be a sequence of random variables such that  $\xi_n$  is independent of  $X_{n+1}, X_{n+2}, \dots$  for each  $n$ . And let  $A = A(t; \lambda)$ ,  $\lambda \in \Lambda$ , be a family of boundary functions, where  $\Lambda$  is an index set. Define

$$(1.1) \quad T = T_\lambda = \inf\{n \geq 1: Z_n > A(n; \lambda)\}, \quad \inf \emptyset = \infty, \text{ for each } \lambda \in \Lambda,$$

where  $Z_n = S_n + \xi_n$  and  $S_n = X_1 + \dots + X_n$  for each  $n$ . This paper concerns the approximations of the distribution of the overshoot, the expected stopping time  $ET$ , the variance of the stopping time  $\text{Var}(T)$  and other quantities related to the stopping time  $T$  as the boundary tends to infinity.

Nonlinear renewal theory concerning boundary crossing times has been studied by Chow (1966), Chow, Hsiung and Lai (1979), Gut (1974), Hagwood (1980), Hagwood and Woodroffe (1982), Lai and Siegmund (1977, 1979), Lalley (1980), Siegmund (1967, 1969) and Woodroffe (1976, 1977), among others.

Lai and Siegmund (1977) studied the limiting distribution of the overshoot for the constant boundary case. In Section 2 we give sufficient conditions for the weak convergence of the overshoot, which generalize the main results of Lai and Siegmund (1977). For the expansion of  $ET_\lambda$  up to  $o(1)$ , different sufficient conditions were given by Woodroffe (1976, 1977) for  $\xi_n = 0$  and by Lai and Siegmund (1979) and Hagwood and Woodroffe (1982) for  $A(t; \lambda) = \text{constant}$ . Their conditions are unified, weakened and generalized to stopping times of form (1.1) in Section 3. Lai and Siegmund (1979) also pointed out that the validity of the expansion of  $\text{Var}(T)$  up to  $o(1)$  was unknown for any special case that the boundary  $A(t; \lambda)$  is nonlinear or  $\xi_n \neq 0$ . In Section 4 the expansion of  $\text{Var}(T)$  is obtained under reasonable conditions.

Define

$$(1.2) \quad \tau(c, u) = \inf\{n \geq 1: S_n - un > c\}, \quad c \geq 0, u \leq \mu.$$

Unlike previous authors, we first consider the difference between  $T_\lambda$  and  $\tau_\lambda = \tau(c_\lambda, d_\lambda)$  for suitable  $c_\lambda$  and  $d_\lambda$ , establish the uniform integrability of  $|T_\lambda - \tau_\lambda|^p$

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(in Section 2) and then derive nonlinear renewal theorems directly from parallel results in the linear case by making use of the uniform integrabilities and the weak convergence of the overshoot. As a result, our theorems are uniform with respect to the drift  $\mu - d_\lambda$  of the random walk  $S_n - d_\lambda n$ , which defines  $\tau$ .

We shall also study the approximations of the expectations of

$$(1.3) \quad U = U_\lambda = 1 + \sum_{n=1}^{\infty} I\{Z_n \leq A(n; \lambda)\},$$

$$(1.4) \quad N = N_\lambda = 1 + \sup\{n \geq 1: Z_n \leq A(n; \lambda)\}, \quad \sup \emptyset = 0,$$

and Blackwell-type renewal theorems relative to the boundary  $A(t; \lambda)$  and the process  $Z_n$ . A number of examples are given in Section 5.

**2. Uniform integrabilities and weak convergences.** Let us define

$$(2.1) \quad b = b_\lambda = \sup\{t \geq 1: A(t; \lambda) \geq \mu t\}, \quad \sup \emptyset = 1,$$

$$(2.2) \quad d = d_\lambda = (\partial A / \partial t)(b_\lambda; \lambda),$$

$$(2.3) \quad \bar{d} = \sup\{(\partial A / \partial t)(t; \lambda): t \geq b_\lambda, \lambda \in \Lambda\},$$

$$(2.4) \quad R = R_\lambda = Z_T - A(T; \lambda),$$

$$(2.5) \quad R(c, u) = S_{\tau(c, u)} - u\tau(c, u) - c, \quad u \leq \mu, c \geq 0,$$

$$(2.6) \quad r(u) = ER^2(0, u) / (2ER(0, u)), \quad u \leq \mu,$$

$$(2.7) \quad G(x, u) = \int_x^\infty P\{R(0, u) > y\} dy / ER(0, u), \quad u \leq \mu, x \geq 0,$$

where  $\tau(c, u)$  is defined by (1.2).

We shall always assume that  $A(t; \lambda)$  is twice differentiable in  $t$  and  $b_\lambda$  is finite so that  $d$  and  $\bar{d}$  are well defined. Our first theorem is a Blackwell-type nonlinear renewal theorem. A random variable  $Y$  has an arithmetic distribution if  $cY$  is integer-valued for some constant  $c \neq 0$ .

**THEOREM 1.** *Let  $R = R_\lambda$  be defined by (2.4). Suppose there exist functions  $\rho(\delta) > 0$ ,  $b^{1/2} \leq \gamma(b) \leq b$ ,  $\gamma(b)/b \rightarrow 0$  as  $b \rightarrow \infty$ , and constant  $d^* < \mu = EX \in (0, \infty)$  such that*

$$(2.8) \quad (T_\lambda - b_\lambda) / \gamma(b_\lambda) = O_P(1), \quad \text{as } b_\lambda \rightarrow \infty,$$

$$(2.9) \quad \lim_n P\left\{\max_{1 \leq j \leq \rho(\delta)\gamma(n)} |\xi_{n+j} - \xi_n| \geq \delta\right\} = 0, \quad \text{for any } \delta > 0,$$

$$(2.10) \quad \sup\{|\gamma^2(b)(\partial^2 A / \partial t^2)(t; \lambda)|: |t - b| \leq K\gamma(b), \lambda \in \Lambda\} < \infty, \forall K,$$

and

$$\lim_{b \rightarrow \infty} d_\lambda = d^*.$$

If  $X - d^*$  does not have an arithmetic distribution, then

$$(2.11) \quad P\{R_\lambda > x\} = G(x) + o(1), \quad \text{as } b_\lambda \rightarrow \infty \text{ for any } x \geq 0,$$

where  $G(x) = G(x, d^*)$  and  $G(x, u)$  is defined by (2.7). If, in addition,  $(T - b)/\gamma(b)$  converges in distribution to a random variable  $W$  as  $b \rightarrow \infty$ , then

$$(2.12) \quad \lim_b P\{R_\lambda > x, T_\lambda \geq b_\lambda + t\gamma(b_\lambda)\} = G(x)P\{W \geq t\},$$

for every real number  $t$  with  $P\{W = t\} = 0$ .

REMARKS. Since  $b^{-\alpha}(T - b) = o_P(1)$  implies (2.8) with  $\gamma(b)/b^\alpha \rightarrow 0$ , Theorem 1 implies the main results of Lai and Siegmund (1977). For the special case  $\gamma(b) = b^{1/2}$ , sufficient conditions for the asymptotic normality of  $T$  are given by Proposition 1 at the end of the section.

PROOF. We shall first fix  $\theta > 0$  and  $t$  real. Let  $m(b)$  be an integer-valued function and  $n_j = n_j(\lambda)$ ,  $j = 0, \dots, k$ , be integers such that

$$(2.13) \quad [\theta\gamma(b)] \leq n_j - n_{j-1} \leq 2[\theta\gamma(b)], \quad 1 \leq j \leq k, \quad n_0 = [b + t\gamma(b)],$$

where  $[x]$  is the integer part of  $x$ ,

$$(2.14) \quad P\{n_j \leq T_\lambda \leq n_j + 2m(b_\lambda)\} \rightarrow 0, \quad \text{as } b_\lambda \rightarrow \infty,$$

and  $0 < m(b) \rightarrow \infty$ ,  $[m(b)/\gamma(b) \rightarrow 0]$ . Define

$$(2.15) \quad \begin{aligned} \bar{d}_j &= [A(n_j; \lambda) - A(n_{j-1}; \lambda)] / (n_j - n_{j-1}), \\ A'_j(n; \lambda, v) &= A(n; \lambda) + v, \quad n < n_{j-1}, \\ &= A(n_{j-1}; \lambda) + (n - n_{j-1})\bar{d}_j + v, \quad n \geq n_{j-1}, \\ \xi'_n &= \xi'_{n,j} = \xi_{\min(n, n_{j-1}-1)} + vI\{n < n_{j-1}\}, \\ Z'_n &= Z'_{n,j} = S_n + \xi'_{n,j}, \\ T'_j &= T'_j(v) = \inf\{n: Z'_{n,j} > A'_j(n; \lambda, v)\}, \end{aligned}$$

and

$$R'_j = R'_j(v) = Z'_{T'_j} - A'_j(T'_j; \lambda, v).$$

Then, by (2.10) and (2.13) for  $n_{j-1} \leq n \leq n_j$ ,

$$|A(n; \lambda) - [A(n_{j-1}; \lambda) + (n - n_{j-1})\bar{d}_j]| \leq \theta^2 M,$$

for some constant  $M$  which does not depend on  $\theta$  and  $\lambda$ , and we can choose  $\theta > 0$  such that

$$(2.16) \quad \delta/2 + A'_j(n; \lambda, -\delta) < A(n; \lambda) < A'_j(n; \lambda, -\delta) + 2\delta,$$

$$(2.17) \quad A'_j(n; \lambda, \delta) - 2\delta < A(n; \lambda) < A'_j(n; \lambda, \delta) - \delta/2,$$

for  $n_{j-1} \leq n \leq n_j$ . Setting  $\theta < \rho(\delta/2)/2$  and  $B_j = \{\max_{n_{j-1} \leq n \leq n_j} |\xi_n - \xi_{n_{j-1}}| >$

$\delta/2\}$ , we have by (2.9), (2.13) and (2.15)–(2.17) for  $n \geq n_{j-1}$  and  $x > 3\delta$ ,

$$(2.18) \quad P\{B_j\} \rightarrow 0, \quad \text{as } b \rightarrow \infty, \quad j = 1, \dots, k,$$

$$(2.19) \quad T'_j(-\delta) \leq T, \quad \text{on } B_j^c \cap \{n_{j-1} \leq T \leq n_j \text{ or } n_{j-1} \leq T'_j(-\delta) \leq n_j\},$$

$$(2.20) \quad T \leq T'_j(\delta), \quad \text{on } B_j^c \cap \{n_{j-1} \leq T \leq n_j \text{ or } n_{j-1} \leq T'(\delta) \leq n\},$$

$$(2.21) \quad \begin{aligned} &P\{R'_j(-\delta) > x, n \leq T'_j(-\delta) \leq n_j\} \\ &\leq P\{R > x - 3\delta, n \leq T \leq n_j\} + o(1) \end{aligned}$$

and

$$(2.22) \quad \begin{aligned} &P\{R > x, n \leq T \leq n_j\} \\ &\leq P\{R'_j(\delta) > x - 3\delta, n \leq T'_j(\delta) \leq n_j\} + o(1). \end{aligned}$$

For  $n \geq n_{j-1}$ , we have

$$(2.23) \quad \begin{aligned} &P\{R'_j > x, T'_j \geq n\} \\ &= \int_0^\infty P\{R(y, \bar{d}_j) > x\} dP\{T'_j \geq n, A'_j(n-1; \lambda, v) - Z'_{n-1} \leq y\}. \end{aligned}$$

Since  $m(b) \rightarrow \infty$  and  $S_n$  is the sum of i.i.d. nondegenerate random variables,

$$\begin{aligned} &\lim_b P\{T'_j \geq n, 0 \leq A'_j(n-1; \lambda, v) - Z'_{n-1} \leq c\}, \quad n \geq m(b) + n_{j-1}, \\ &\leq \lim_b \sup_y P\{y \leq S_{m(b)} \leq y + c\} = 0, \quad \text{for every } c. \end{aligned}$$

It follows from (2.23) and Theorem 7 stated at the end of Section 4 that

$$(2.24) \quad \begin{aligned} &P\{R'_j > x, T'_j \geq n\} = G(x)P\{T'_j \geq n\} + o(1), \\ &n \geq m(b) + n_{j-1}. \end{aligned}$$

By (2.14), (2.18)–(2.20) and (1.2)

$$(2.25) \quad \begin{aligned} &P\{n_{j-1} \leq T'_j(-\delta) \leq n_{j-1} + m(b)\} \\ &\leq P\{B_j\} + P\{n_{j-1} \leq T \leq n_{j-1} + 2m(b)\} \\ &\quad + P\{T'_j(\delta) - T'_j(-\delta) \geq m(b)\} \\ &= o(1) + P\{\tau(2\delta, \bar{d}_j) \geq m(b)\} = o(1) \end{aligned}$$

and

$$(2.26) \quad \begin{aligned} &P\{n_{j-1} \leq T'_j(\delta) \leq n_j\} \leq P\{n_{j-1} \leq T \leq n_j\} + o(1) \\ &\leq P\{n_{j-1} \leq T'_j(-\delta) \leq n_j\} + o(1). \end{aligned}$$

Therefore by (2.21)–(2.22) and (2.24)–(2.26) we have for  $x > 3\delta$ ,

$$\begin{aligned}
 (2.27) \quad & G(x + 3\delta)P\{n_{j-1} \leq T \leq n_j\} \\
 & \leq G(x + 3\delta)P\{n_{j-1} + m(b) \leq T'_j(-\delta) \leq n_j\} + o(1) \\
 & \leq P\{R'_j(-\delta) > x + 3\delta, n_{j-1} \leq T'_j(-\delta) \leq n_j\} + o(1) \\
 & \leq P\{R > x, n_{j-1} \leq T \leq n_j\} + o(1) \\
 & \leq G(x - 3\delta)P\{n_{j-1} \leq T \leq n_j\} + o(1),
 \end{aligned}$$

which implies by summing up over  $j$  that

$$\begin{aligned}
 (2.28) \quad & G(x + 3\delta)P\{n_0 \leq T \leq n_k\} \leq P\{R > x, n_0 \leq T \leq n_k\} + o(1) \\
 & \leq G(x - 3\delta)P\{n_0 \leq T \leq n_k\} + o(1).
 \end{aligned}$$

Hence the proof is complete by taking limits in the order  $b \rightarrow \infty$ ,  $k \rightarrow \infty$  [ $t \rightarrow -\infty$  for (2.11)] and then  $\delta \rightarrow 0$ .  $\square$

To study the uniform integrabilities of the powers of the differences of linear and nonlinear stopping times, we shall first give the regularity conditions on  $\xi$ . The process  $\xi = \{\xi_n, n \geq 1\}$  is said to be regular with  $p \geq 0$  and  $1/2 < \alpha \leq 1$  if there exists a random variable  $L$ , a function  $f(x)$  and a sequence of random variables  $V_n$ ,  $n \geq 1$ , such that

$$(2.29) \quad \xi_n = f(n) + V_n, \quad \text{for } n \geq L \text{ and } EL^p < \infty,$$

$$(2.30) \quad \max_{1 \leq j \leq \sqrt{n}} |f(n+j) - f(n)| \leq K, \quad K < \infty,$$

$$(2.31) \quad \left\{ \max_{1 \leq j \leq n^\alpha} |V_{n+j}|^p, n \geq 1 \right\} \text{ is uniformly integrable,}$$

$$(2.32) \quad n^p P\left\{ \max_{0 \leq j \leq n} V_{n+j} \geq \theta n^\alpha \right\} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \forall \theta > 0,$$

and for some  $w > 0$ ,  $w < \mu - \bar{d}$  if  $\alpha = 1$ ,

$$(2.33) \quad \sum_{n=1}^{\infty} n^{p-1} P\{-V_n \geq wn^\alpha\} < \infty.$$

The sequential procedures that motivate the decomposition (2.29) have been discussed by Pollak and Siegmund (1975) and Lai and Siegmund (1977). In Proposition 1 (2.31)–(2.33) are replaced by a single-moment condition in some special cases.

We shall set  $f(n)$  to be the median of  $\xi_n$  when  $\xi$  is not regular and extend  $f$  to a function on  $[1, \infty)$  by linear interpolation. Therefore we can define

$$(2.34) \quad \tau = \tau_\lambda = \tau(c_\lambda, d_\lambda),$$

where  $\tau(c, u)$  is defined by (1.2) and  $c_\lambda = b_\lambda(\mu - d_\lambda) - f(b_\lambda)$ . The following theorem will be proved in Section 6.

**THEOREM 2.** Suppose that  $\xi$  is regular with  $p \geq 1$ ,  $1/2 < \alpha \leq 1$ ,  $E|X|^{(p+1)/\alpha} < \infty$  and that there exist constants  $\delta$  and  $\mu^*$  with  $0 < \delta < 1$  and  $\mu^* < \mu$  such that

$$(2.35) \quad b^p P\{T \leq \delta b\} \rightarrow 0, \quad \text{as } b \rightarrow \infty,$$

and

$$(2.36) \quad (\partial A / \partial t)(t; \lambda) \leq \mu^*, \quad t \geq \delta b, \lambda \in \Lambda.$$

(i) If  $E|X|^{2p} < \infty$  and

$$(2.37) \quad \sup \left\{ |b_\lambda(\partial^2 A / \partial t^2)(t; \lambda)| : b_\lambda - Kb_\lambda^\alpha \leq t \leq b_\lambda + Kb_\lambda^\alpha, \lambda \in \Lambda \right\} < \infty,$$

for any  $K > 0$ , then

$$(2.38) \quad \{|T_\lambda - \tau_\lambda|^p : \lambda \in \Lambda\} \text{ is uniformly integrable.}$$

(ii) (2.38) still holds without the condition that  $E|X|^{2p} < \infty$ , if  $\partial^2 A / \partial t^2 = 0$  on  $t \geq \delta b$ .

(iii) If, in addition to the conditions in (i) or in (ii), the condition (2.33) is strengthened to

$$(2.39) \quad \sum_{n=1}^{\infty} n^{p-1} P \left\{ \sup_{j \geq n} -j^{-\alpha} V_j > w \right\} < \infty,$$

for some  $w > 0$ ,  $w < \mu - \bar{d}$  if  $\alpha = 1$ , then

$$(2.40) \quad \{|N_\lambda - T_\lambda|^p : \lambda \in \Lambda\} \text{ is uniformly integrable}$$

and

$$(2.41) \quad \{|U_\lambda - T_\lambda|^p : \lambda \in \Lambda\} \text{ is uniformly integrable.}$$

**COROLLARY 1.** (i) Suppose that the conditions of Theorem 2(i) or (ii) hold. Then

$$(2.42) \quad ET_\lambda = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + O(1), \quad \text{if } p = 1,$$

$$(2.43) \quad \text{Var}(T_\lambda) = (\mu - d_\lambda)^{-2} \sigma^2 b_\lambda + b_\lambda^{1/2} O(1), \quad \text{if } p = 2.$$

(ii) Suppose that the conditions of Theorem 2(iii) hold. Then

$$(2.44) \quad EN_\lambda \text{ (and } EU_\lambda) = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + O(1), \quad \text{if } p = 1,$$

$$(2.45) \quad \text{Var}(N_\lambda) [\text{and } \text{Var}(U_\lambda)] = (\mu - d_\lambda)^{-2} \sigma^2 b_\lambda + b_\lambda^{1/2} O(1), \quad \text{if } p = 2.$$

**PROOF.** It follows from (2.30) that

$$(2.46) \quad f(b) = b^{1/2} O(1).$$

Therefore

$$E\tau_\lambda = (\mu - d_\lambda)^{-1} c_\lambda + O(1) = b_\lambda - (\mu - d_\lambda)^{-1} f(b_\lambda) + O(1)$$

and

$$\text{Var}(\tau_\lambda) = (\mu - d_\lambda)^{-3} \sigma^2 c_\lambda + c_\lambda^{1/2} O(1) = (\mu - d_\lambda)^{-2} \sigma^2 b_\lambda + b_\lambda^{1/2} O(1). \quad \square$$

REMARKS. The stopping time  $\tau_\lambda$  can be written as

$$(2.47) \quad \tau_\lambda = \inf\{n \geq 1: S_n + f(b_\lambda) > \mu b_\lambda + d_\lambda(n - b_\lambda)\},$$

where  $\mu b_\lambda + d_\lambda(t - b_\lambda)$  is the two-term Taylor expansion of  $A(t; \lambda)$  in the neighborhood  $(b_\lambda - Kb_\lambda^\alpha, b_\lambda + Kb_\lambda^\alpha)$  in which the residual  $A(t; \lambda) - \mu b_\lambda - d_\lambda(t - b_\lambda)$  is dominated by  $O(1)(t - b_\lambda)^2/b_\lambda$ . Therefore  $T_\lambda$  should be very close to  $\tau_\lambda$  in view of the compactness of  $(T_\lambda - b_\lambda)^2/b_\lambda$  and  $\xi_T - f(b_\lambda)$ . In Sections 3 and 4 we will see that more detailed expansions of  $A(t; \lambda)$  lead to higher-order expansions of quantities relative to  $T$ . The stopping times  $\tau(c, u)$  have been used by Woodroffe and Lai and Siegmund to obtain the expansions of  $ET$  up to  $o(1)$ . But they did not associate each  $T_\lambda$  with a  $\tau_\lambda = \tau(c_\lambda, u_\lambda)$  and consider the difference between them. Also our theorems are uniform in the drift of the random walk in the sense that we can always set  $\lambda' = (\lambda, u)$  and  $A(t; \lambda') = A(t; \lambda) + ut$  so that

$$(2.48) \quad \begin{aligned} T_{\lambda'} &= \inf\{n \geq 1: Z_n - un > A(n; \lambda)\} \\ &= \inf\{n \geq 1: Z_n > A(n; \lambda')\}. \end{aligned}$$

PROPOSITION 1. Let  $T = T_\lambda$  be defined by (1.1) and  $\text{Var}(X) = \sigma^2 < \infty$ .

(i) Suppose that  $\lim A(n; \lambda) = \infty$  as  $b \rightarrow \infty$  for every fixed  $n \geq 1$ ,

$$\liminf_{b \rightarrow \infty} \inf\{A(n; \lambda)/(2n \log_2 n)^{1/2}: n \leq \delta b_\lambda\} > \sigma,$$

and

$$P\left\{\limsup_{n \rightarrow \infty} \xi_n/(n \log_2 n)^{1/2} = 0\right\} = 1.$$

Then  $P\{T_\lambda \leq \delta b_\lambda\} \rightarrow 0$  as  $b_\lambda \rightarrow \infty$ .

(ii) Suppose that  $P\{n^{-1/2}\xi_n \rightarrow 0\} = 1$ ,  $\lim P\{T_\lambda \leq \delta b_\lambda\} = 0$  for any  $0 < \delta < 1$  and for some  $d^* < \mu$ ,

$$(2.49) \quad \lim_{\delta \rightarrow 0} \limsup_{b \rightarrow \infty} \sup\{|\partial A/\partial t(t; \lambda) - d^*|: |t - b_\lambda| \leq \delta b_\lambda\} = 0.$$

Then  $b_\lambda^{-1/2}(T_\lambda - b_\lambda)$  converges in distribution to  $N(0, \sigma^2/(\mu - d^*)^2)$ .

(iii) Let  $Y_1, Y_2, \dots$  be a sequence of i.i.d. random vectors with mean zero. Suppose that  $|V_n| \leq \|Y_1 + \dots + Y_n\|^2/n$  for each  $n$  and  $E\|Y\|^{2p/\alpha} < \infty$  with  $p \geq 1$  and  $1/2 < \alpha \leq 1$ . Then (2.31)–(2.33) and (2.39) hold.

PROOF. (i) follows from the law of the iterated logarithm. (iii) follows from Chow and Lai (1978), Theorem 1, and Doob's inequalities for submartingales. (ii) may be proved as follows:

$$(2.50) \quad \begin{aligned} P\{T > (1 + \delta)b\} &\leq P\left\{\max_{n \leq (1+\delta)b} Z_n \leq \mu b + (d^* + o(1))\delta b\right\} \\ &= o(1), \text{ by SLLN.} \end{aligned}$$

By the conditions given and (2.50)

$$(2.51) \quad T_\lambda/b_\lambda \rightarrow 1, \quad \xi_T/b_\lambda^{1/2} \rightarrow 0, \quad (A(T; \lambda) - \mu b_\lambda)/(T_\lambda - b_\lambda) \rightarrow d^*,$$

in probability as  $b_\lambda \rightarrow \infty$ . By the definitions (1.1) and (2.4)

$$(2.52) \quad \begin{aligned} S_T - \mu T &= R + A(T; \lambda) - \mu b + \mu(b - T) - \xi_T \\ &= R + (\mu - d^* + o_P(1))(b - T) + o_P(1)b^{1/2}. \end{aligned}$$

Since (2.51) still holds when  $T$  is replaced by  $T - 1$  and  $EX_T^2 = o(1)b$ ,

$$(2.53) \quad R = o_P(1)b^{1/2}, \quad \text{as } b \rightarrow \infty, \text{ where } T' = \min(T, 2b).$$

Therefore by (2.52) and (2.53)

$$(S_T - \mu T)/b^{1/2} = (\mu - d^* + o_P(1))(b - T)/b^{1/2} + o_P(1).$$

By Anscombe's theorem and Slutsky's theorem the proof is complete.  $\square$

**3. Expansions of  $ET$ ,  $EU$  and  $EN$ .** The following theorem is obviously a consequence of Theorems 1 and 2.

**THEOREM 3.** *Let  $T$ ,  $U$  and  $N$  be defined by (1.1), (1.3) and (1.4). Suppose that (2.9) holds with  $\gamma(n) = n^{1/2}$  and  $\rho(\delta) = 1$ , (2.29) holds with  $p = 0$  and that there exist constants  $d_1^* < \mu$  and  $d_2^*$  such that*

$$(3.1) \quad \lim_{n \rightarrow \infty} \max_{0 \leq j \leq \sqrt{n}} |f(n+j) - f(n)| = 0,$$

$$(3.2) \quad V_n \text{ converges in distribution to an integrable random variable } V,$$

$$(3.3) \quad \lim_{b \rightarrow \infty} d_\lambda = d_1^* \text{ and } X - d_1^* \text{ is nonarithmetic, and}$$

$$(3.4) \quad \lim_{b \rightarrow \infty} \sup \left\{ |b_\lambda(\partial^2 A / \partial t^2)(t; \lambda) - d_2^*| : (t - b_\lambda)^2 \leq Kb_\lambda \right\} = 0,$$

for any constant  $K$ .

(i) If  $\text{Var}(X) = \sigma^2 < \infty$  and (2.38) holds for  $p = 1$ , then

$$(3.5) \quad ET_\lambda = b_\lambda - (\mu - d_\lambda)^{-1}f(b_\lambda) + C_0 + o(1), \quad \text{as } b_\lambda \rightarrow \infty,$$

where

$$C_0 = (\mu - d_1^*)^{-1} \left( r(d_1^*) + (\mu - d_1^*)^{-2} d_2^* \sigma^2 / 2 - EV \right),$$

and  $r(\cdot)$  is defined by (2.6).

(ii) If  $\text{Var}(X) = \sigma^2 < \infty$  and (2.40) holds with  $p = 1$ , then

$$(3.6) \quad \begin{aligned} EN_\lambda &= b_\lambda - (\mu - d_\lambda)^{-1}f(b_\lambda) + C_0 \\ &\quad - \int_0^\infty EN(-x, d_1^*) dG(x) + o(1), \quad \text{as } b_\lambda \rightarrow \infty, \end{aligned}$$

where  $G(x)$  is given in (2.11) and

$$(3.7) \quad N(x, u) = \sup \{n : S_n - un \leq x\}, \quad \sup \emptyset = 0, u < \mu, -\infty < x < \infty.$$

(iii) If  $\text{Var}(X) = \sigma^2 < \infty$  and (2.41) holds with  $p = 1$ , then

$$(3.8) \quad EU_\lambda = b_\lambda - (\mu - d_\lambda)^{-1}f(b_\lambda) + C_1 + o(1), \quad \text{as } b_\lambda \rightarrow \infty,$$

where

$$C_1 = (\mu - d_1^*)^{-2} [\sigma^2 + (\mu - d_1^*)^2 + (\mu - d_1^*)^{-1} d_2^* \sigma^2 - 2(\mu - d_1^*) EV] / 2.$$

COROLLARY 2. Under the conditions of Theorem 3(ii), for  $x > 0$ ,

$$(3.9) \quad \lim_b \sum_{n=1}^{\infty} P\{A(n; \lambda) - x \leq Z_n \leq A(n; \lambda)\} = (\mu - d_1^*)^{-1} x$$

and

$$\lim_b \sum_{n=1}^{\infty} P\{A(n; \lambda) \leq Z_n \leq A(n; \lambda) + x\} = (\mu - d_1^*)^{-1} x.$$

COROLLARY 3. Under the conditions of Theorem 3(ii)

$$(3.10) \quad \begin{aligned} & \lim_b \sum_{n=1}^{\infty} P\{T > n, A(n; \lambda) - Z_n < x\} \\ &= (\mu - d_1^*)^{-1} \int_0^x P\left\{\min_{n \geq 1} (S_n - d_1^* n) \geq -y\right\} dy. \end{aligned}$$

REMARKS. Theorem 3 still holds when  $f(n)$  is the sum of  $cn^{1/2}$  and a term satisfying (3.1), since  $cn^{1/2}$  can be absorbed by  $A(n; \lambda)$ . Theorems 2 and 3 imply Theorem 3 of Lai and Siegmund (1979) and Theorem 2 of Hagwood and Woodroffe (1982).

PROOF OF THEOREM 3. (i) By the definitions

$$S_T = R + A(T; \lambda) - \xi_T$$

and

$$S_\tau = R(c_\lambda, d) + \mu b + d(\tau - b) - f(b).$$

It follows that

$$(3.11) \quad \begin{aligned} & S_T - S_\tau - d(T - \tau) \\ &= R - R(c_\lambda, d) + [A(T; \lambda) - \mu b - d(T - b)] - [\xi_T - f(b)]. \end{aligned}$$

Since

$$\begin{aligned} S_T - \mu T - S_\tau + \mu \tau &= [(S_{\max(T, \tau)} - \mu \max(T, \tau)) - (S_\tau - \mu \tau)] \\ &\quad - [(S_\tau - \mu \tau) - (S_{\min(T, \tau)} - \mu \min(T, \tau))], \end{aligned}$$

it follows from Chow, Robbins and Teicher (1965) that

$$\begin{aligned} E(S_T - S_\tau - \mu(T - \tau))^2 &= \sigma^2 [E(\max(T, \tau) - \tau) + E(\tau - \min(T, \tau))] \\ &= \sigma^2 E|T - \tau|. \end{aligned}$$

Therefore by (2.38)

$$(3.12) \quad S_T - S_\tau - d(T - \tau) \text{ is uniformly integrable.}$$

By Anscombe's central limit theorem  $(\tau - b)b^{-1/2}$  converges in distribution to  $(\mu - d_1^*)^{-1}\sigma N(0, 1)$ . It follows that, as  $b \rightarrow \infty$ ,

$$(3.13) \quad (T - b)b^{1/2} \rightarrow \sigma^* N(0, 1), \quad \text{in distribution, by (2.38),}$$

$$(3.14) \quad \lim P\{R > x\} = \lim P\{R(c, d) > x\} = G(x), \quad \text{by Theorem 1,}$$

$$(3.15) \quad A(T; \lambda) - \mu b - d(T - b) \rightarrow d_2^*(\sigma^* N(0, 1))^2/2,$$

in distribution, by (3.4),

$$(3.16) \quad \xi_T - f(b) \rightarrow V, \quad \text{in distribution, by (2.9), (3.1) and (3.2),}$$

where  $c = c_\lambda = (\mu - d_\lambda)b_\lambda - f(b_\lambda)$  and  $\sigma^* = (\mu - d_1^*)^{-1}\sigma$ .

It follows from a simple renewal argument that  $R(c_\lambda, d_\lambda)$  is uniformly integrable. Hence by (3.11)–(3.16)

$$E(S_T - S_\tau - d(T - \tau)) = (\mu - d_1^*)^{-2}\sigma^2 d_2^*/2 - EV + o(1),$$

$$ET = E\tau + (\mu - d_1^*)^{-3}\sigma^2 d_2^*/2 - (\mu - d_1^*)^{-1}EV + o(1),$$

and

$$\begin{aligned} E\tau &= (\mu - d)^{-1}c + (\mu - d)^{-1}r(d^*) + o(1) \\ &= b - (\mu - d)^{-1}f(b) + (\mu - d_1^*)^{-1}r(d_1^*) + o(1). \end{aligned}$$

This completes the proof of (i).

(ii) Define for  $v < \mu - d_1^*$ ,

$$(3.17) \quad N'_\lambda(v) = \sup\{n: \xi_T + S_{T+n} \leq A(T; \lambda) + (d_1^* + v)n\}, \quad \sup \emptyset = 0.$$

Then by (2.9), (3.3), (3.4), (3.13), (2.40) and (3.7)

$$(3.18) \quad \lim_b P\{N'_\lambda(-v) \leq N - T \leq N'_\lambda(v)\} = 1, \quad \text{for every } v > 0,$$

$$(3.19) \quad P\{N'_\lambda(v) \leq m\} = \int_0^\infty P\{N(-x, d_1^* + v) \leq m\} dP\{R \leq x\}.$$

By (3.7)  $N(x, u)$  is monotone in  $x$  and in  $u$  and continuous in  $u$  at  $d_1^*$  a.s. when  $\sum_{n=1}^\infty P\{S_n - d_1^* n = x\} = 0$ . Since  $G(x)$  has a density,

$$(3.20) \quad \lim_{v \rightarrow 0} \lim_b P\{N'_\lambda(v) \leq m\} = \int_0^\infty P\{N(-x, d_1^*) \leq m\} d(1 - G(x)).$$

Therefore by (3.18)–(3.20) the limiting distribution of  $N - T$  is given by the right-hand side of (3.20) and (3.6) follows from the uniform integrability of  $N - T$ . The proof of (iii) is similar and omitted.  $\square$

**PROOF OF COROLLARIES 2 AND 3.** Let  $\lambda' = (\lambda, x)$  and  $A(t; \lambda') = A(t; \lambda) - x$ . Then  $b_{\lambda'} = b_\lambda - (\mu - d_1^*)^{-1}x + o(1)$ ,  $d_{\lambda'} = d_\lambda + b_\lambda^{-1}O(1)$  and  $f(b_{\lambda'}) = f(b_\lambda) + o(1)$ . Therefore (3.9) follows from (3.8). For (3.10) we shall write (3.10) in two

parts,

$$\begin{aligned} & \sum_{n=1}^{\infty} P\{T > n, A(n; \lambda) - Z_n < x\} \\ &= \sum_{n=1}^{\infty} P\{A(n; \lambda) - x < Z_n \leq A(n; \lambda)\} \\ &\quad - \sum_{n=1}^{\infty} P\{T \leq n, A(n; \lambda) - x < Z_n \leq A(n; \lambda)\}. \end{aligned}$$

The limit of the first term is given by (3.9) and the techniques in the proof of Theorem 3(ii) bring the second term to the linear case. The final formulation is taken from Woodroffe (1976), Theorem 3.1.  $\square$

**4. Expansion of  $\text{Var}(T)$ .** We shall study the case

$$(4.1) \quad T = T_\lambda = \inf\{n \geq 1: S_n > A(n; \lambda)\}, \quad \inf \emptyset = \infty,$$

where  $S_n = X_1 + \cdots + X_n$  and  $X, X_1, X_2, \dots$  is a sequence of i.i.d. random variables with  $EX = \mu > 0$  and  $0 < \text{Var}(X) = \sigma^2 < \infty$ .

For  $A(t; \lambda) = \lambda t^\alpha$ ,  $0 \leq \alpha < 1$ , the first term in the expansion

$$\text{Var}(T) = (1 - \alpha)^{-2} (\sigma/\mu)^2 (\lambda/\mu)^{1/(1-\alpha)} (1 + o(1)) \quad \text{as } \lambda \rightarrow \infty,$$

was obtained by Woodroffe (1976) and Chow, Hsiung and Lai (1979) under certain moment conditions on  $X$ . For the linear case that  $A(t; \lambda) = \lambda = \text{constant}$ , the second-order expansion of  $\text{Var}(T)$ ,

$$(4.2) \quad \text{Var}(T) = \mu^{-3} \sigma^2 \lambda + \mu^{-2} C_1^* + o(1) \quad \text{as } \lambda \rightarrow \infty,$$

was obtained by Smith (1959) for  $P\{X < 0\} = 0$  and by Lai and Siegmund (1979) for  $P\{X < 0\} > 0$ . The variances of boundary crossing times have also been studied by Chow (1966), Siegmund (1969), Gut (1974) and Lai (1975). The following theorem gives the expansion of  $\text{Var}(T)$  up to  $o(1)$  for the general case.

**THEOREM 4.** *Let  $T$  be defined by (4.1). Suppose that for some constants  $0 < \delta < 1$ ,  $\mu^* < \mu$ ,  $d_1^*$ ,  $d_2^*$  and  $d_3^*$ ,*

$$(4.3) \quad \lim_{b \rightarrow \infty} b_\lambda^2 P\{T_\lambda \leq \delta b_\lambda\} = 0, \quad 0 < \delta < 1,$$

$$(4.4) \quad (\partial A / \partial t)(t; \lambda) \leq \mu^* < \mu, \quad \text{for any } t \geq \delta b_\lambda \text{ and } \lambda \in \Lambda,$$

$$(4.5) \quad \lim d_\lambda = d_1^* \quad \text{and} \quad \lim b_\lambda (\partial^2 A / \partial t^2)(b_\lambda; \lambda) = d_2^*, \quad \text{as } b_\lambda \rightarrow \infty,$$

$$(4.6) \quad \sup \{b_\lambda^2 |(\partial^3 A / \partial t^3)(t; \lambda)| : |t - b_\lambda| \leq \theta b, \lambda \in \Lambda\} < \infty, \quad \text{some } \theta > 0,$$

$$(4.7) \quad \limsup_b \{b_\lambda^2 |(\partial^3 A / \partial t^3)(t; \lambda) - d_3^*| : (t - b)^2 < Kb\} = 0, \quad \text{any } K,$$

where  $b = b_\lambda$  and  $d = d_\lambda$  are defined by (2.1) and (2.2).

If  $X$  has a density  $p(x)$  with respect to Lebesgue measure  $dx$  such that

$$(4.8) \quad \sum_{n=-\infty}^{\infty} \sup_{n \leq x < n+1} p(x) < \infty \quad \text{and} \quad EX^4 < \infty,$$

then

$$(4.9) \quad \text{Var}(T_\lambda) = (\mu - d_\lambda)^{-2} \sigma^2 b_\lambda + \mu_0^{-2} C^* + o(1), \quad \text{as } b_\lambda \rightarrow \infty,$$

where  $\mu_0 = \mu - d_1^*$  and  $C^*$  can be written as

$$(4.10) \quad \begin{aligned} C^* = & C_1^* - 2\mu_0^{-2} d_2^* \sigma^2 C_2^* - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3} d_3^* \sigma^4 \\ & + \mu_0^{-4} d_2^* \sigma^4 (7d_2^* + 11\mu_0)/2, \end{aligned}$$

with

$$\begin{aligned} C_1^* = & \sigma^2 r(d_1^*)/\mu_0 + 3(r(d_1^*))^2 + 2 \int_0^\infty x^2 dG(x, d_1^*) \\ & + 2r(d_1^*) E \max_{n \geq 0} (nd_1^* - S_n)^+ \\ & - 2 \int_0^\infty ER(x, d_1^*) P\left\{ \max_{n \geq 0} (nd_1^* - S_n) \geq x \right\} dx, \\ C_2^* = & \mu_0 \sum_{n=1}^\infty P\{S_n - nd_1^* \leq 0\} + \sum_{n=1}^\infty n^{-1} E(S_n - nd_1^*)^-, \end{aligned}$$

and  $R(\cdot, \cdot)$ ,  $r(\cdot)$  and  $G(\cdot, \cdot)$  given by (2.5)–(2.7).

REMARKS. When  $A(t; \lambda) = c_\lambda + td_1^*$ ,  $d_2^* = d_3^* = 0$  and  $C^*$  agrees with  $C_1^*$ , which has the same expression as the constant in Theorem 5 of Lai and Siegmund (1979). Keener (1987) pointed out that  $C_1^*$  can be written in terms of moments of ladder variables. Usually, (4.3) can be verified by Lemma 5 of Chow and Lai (1978). But a special argument is often required for each family of boundary curves so that the lemma can be used.

We shall split the proof of Theorem 4 into several lemmas. Under the conditions of Theorem 4, our lemmas also imply the following statements:

$$(4.11) \quad \begin{aligned} E(T_\lambda - b_\lambda)^3/b_\lambda = & \mu_0^{-3} [3\sigma^2 r(d_1^*) - E(X - \mu)^3] \\ & + 3\mu_0^{-4} \sigma^4 + 9\mu_0^{-5} d_2^* \sigma^4/2 + o(1), \quad \text{as } b_\lambda \rightarrow \infty. \end{aligned}$$

Under the further assumption that

$$(4.12) \quad \inf\{A(t; \lambda)/t^2: t > b_\lambda, \lambda \in \Lambda\} > -\infty,$$

$$(4.13) \quad \begin{aligned} \text{Cov}(T_\lambda, R_\lambda) = & \mu_0^{-1} \int_0^\infty [ER(x, d_1^*) - r(d_1^*)] P\left\{ \max_n (S_n - nd_1^*)^- \leq x \right\} dx \\ & + (\sigma/\mu_0)^2 d_2^* \left[ \left(1 - (\sigma/\mu_0)^2\right)/2 + \sum_{n=1}^\infty P\{S_n - nd_1^* \leq 0\} \right] \\ & + o(1), \quad \text{as } b_\lambda \rightarrow \infty. \end{aligned}$$

The proof of Theorem 4 is contained in Section 7 where the linear renewal theorems in Zhang (1986) are used. We shall state these results here for reference.

**THEOREM 5.** *Let  $R(c, u)$  and  $r(u)$  be defined by (2.5) and (2.6). Suppose that  $X$  is strongly nonlattice in the sense of Stone (1965) with  $E|X|^3 < \infty$  and that for some  $k < \infty$  and  $-\infty < a \leq b < \mu$ ,*

$$(4.14) \quad \sup \left\{ \int_{-\infty}^{\infty} |E \exp[itR(0, u)]|^k dt : a \leq u \leq b \right\} < \infty, \quad i^2 = -1.$$

*Then*

$$(4.15) \quad \int_0^{\infty} \sup \{ |ER(y, u) - r(u)| : y \geq x, a \leq u \leq b \} dx < \infty.$$

**THEOREM 6.** *Let  $R(c, u)$  and  $r(u)$  be defined by (2.5) and (2.6). Suppose that  $X$  has a density  $p(x)$  such that (4.8) holds. Then (4.14) holds and for any  $-\infty < a \leq b < \mu$ ,*

$$(4.16) \quad \sum_{n=0}^{\infty} \sup \{ |(\partial/\partial u)(ER(x, u) - r(u))| : n \leq x < n+1, a \leq u \leq b \} < \infty.$$

**THEOREM 7.** *Let  $R(c, u)$  and  $G(x, u)$  be defined by (2.5) and (2.7), respectively. Assume that  $c_n$  and  $u_n$  are two sequences of constants such that  $c_n \rightarrow \infty$  and  $u_n \rightarrow u^* < \mu$  as  $n \rightarrow \infty$ . If  $X - u^*$  does not have an arithmetic distribution, then*

$$(4.17) \quad P\{R(c_n, u_n) > x\} \rightarrow G(x, u^*), \quad \text{as } n \rightarrow \infty.$$

**REMARK.** The existence of  $EX$  is assumed in Section 1.

Basically, Theorems 5 and 6 can be proved by the Fourier method.

**5. Discussion and examples.** We shall first compare the results in this paper with previous studies. To fix the ideas, let us consider the expectations of stopping times which can be written as

$$(5.1) \quad T_\lambda = \inf\{n : S_n > A(n; \lambda)\}$$

$$(5.2) \quad = \inf\{n : ng(S_n/n) + f(n) > c\}$$

$$(5.3) \quad = \inf\{n : S_n - nd^* + \xi_n > c\}, \quad c = c_\lambda,$$

for some smooth functions  $g(t)$  and  $f(t)$ , where  $S_n$  is a random walk. Woodrooffe (1976, 1977) considered the case that  $A(t; \lambda) = \lambda a(t)$  for a regular varying function  $a(t)$ , took the Taylor expansion of  $a(t)$  and used a local limit theorem to obtain the expansion of  $ET_\lambda$ . Lai and Siegmund (1979) took the Taylor expansion of  $g(t)$ , rewrote  $T_\lambda$  into the form (5.3) and used the classical (linear) renewal theory to obtain the expansion of  $ET_\lambda$ . In this paper, linear renewal

theorems with varying drift are used to derive the expansion of  $ET_\lambda$ . Since better linear renewal theorems are used, we can take Taylor expansions of either  $A(t; \lambda)$  or  $g(t)$  and we do not need the help of local limit theorems. As a result, our theorems give a unified treatment for stopping times of a more general form (1.1), require less and weaker regularity conditions and allow the shape of the boundary curve and the slope of the random walk to change. For example, (2.9) used in our Theorem 3 is much weaker than corresponding (3) in Hagwood and Woodroffe (1982) with  $\alpha = 1$  and (17) in Lai and Siegmund (1979) in view of the proof of Proposition 1 in Lai and Siegmund (1979). However, taking Taylor expansions of  $A(t; \lambda)$  has two technical advantages when  $T_\lambda$  can be written as both (5.1) and (5.3). The first is that the regularity conditions with respect to  $A(t; \lambda)$  are often easy to check. The second is that we can include more terms in the Taylor expansion of  $A(t; \lambda)$  to obtain further expansions of quantities relative to the stopping times  $T$  by the techniques in Section 7, whereas the second-order expansion of  $\xi_n$  is almost impossible to describe.

We shall demonstrate the methods of checking the conditions in our theorems by the following examples, which also give comparisons of our results with previous studies. We shall always put  $S_n = X_1 + \cdots + X_n$ , where  $X, X_1, X_2, \dots$  is a sequence of i.i.d. random variables.

**EXAMPLE 1** [Robbins (1970)]. Let  $T = T_\lambda$  be defined by (4.1) with  $EX = \mu > 0$ ,  $0 < \text{Var}(X) = \sigma^2 < \infty$  and

$$(5.4) \quad A(t; \lambda) = (t(\log t + 2\lambda))^{1/2}, \quad \lambda > 0.$$

Then as  $\lambda \rightarrow \infty$ ,

$$(5.5) \quad b_\lambda = \mu^{-2}(\log b_\lambda + 2\lambda) = \mu^{-2}(2\lambda + \log(2\lambda) - 2\log \mu) + o(1),$$

$$(5.6) \quad d_\lambda = \mu/2 + (2\mu b_\lambda)^{-1} \rightarrow d_1^* = \mu/2.$$

It follows from Theorem 2 that

$$(5.7) \quad ET_\lambda = \mu^{-2}(2\lambda + \log(2\lambda)) + O(1),$$

$$(5.8) \quad \text{Var}(T_\lambda) = 8\mu^{-4}\sigma^2\lambda + O(\lambda^{1/2}), \quad \text{if } EX^4 < \infty.$$

Suppose that  $X - \mu/2$  does not have an arithmetic distribution. Then, by Theorems 1–3 we have, with  $s_n = S_n - \mu n/2$  and  $\tau = \inf\{n: s_n > 0\}$ ,

$$(5.9) \quad ET_\lambda = \mu^{-2}[2\lambda + \log(2\lambda) - 2\log \mu - \mu\sigma^2/2 + \mu Es_\tau^2/Es_\tau] + o(1).$$

Furthermore, if  $X$  has a density  $p(x)$  satisfying (4.8), then

$$(5.10) \quad \begin{aligned} \mu^2 \text{Var}(T_\lambda) = & 4\mu^{-2}\sigma^2(2\lambda + \log(2\lambda)) - 8\mu^{-2}\sigma^2\log \mu + 8\mu^{-2}\sigma^2 + 4C_1^* \\ & + 8\mu^{-1}\sigma^2C_2^* + 4\mu^{-1}E(X - \mu)^3 - 18\mu^{-2}\sigma^4 + o(1), \end{aligned}$$

where

$$C_1^* = \mu^{-1} \sigma^2 E s_\tau^2 / E s_\tau + (3/4) (E s_\tau^2 / E s_\tau)^2 - (2/3) E s_\tau^3 / E s_\tau \\ - (E s_\tau^2 / E s_\tau) E \min_{n \geq 0} s_n - 2 \int_0^\infty r^*(x) P\{\min s_n \leq -x\} dx, \\ C_2^* = 2^{-1} \mu \sum_{n=1}^\infty P\{s_n \leq 0\} + \sum_{n=1}^\infty n^{-1} E s_n^-,$$

and

$$r^*(x) = E(s_{\tau(x)} - x), \quad \text{with } \tau(x) = \inf\{n: s_n > x\}.$$

**PROOF.**

$$(5.11) \quad (\partial A / \partial t)(t; \lambda) = A(t; \lambda) / (2t) + 1 / (2A(t; \lambda)).$$

Also, we have

$$\partial^2 A / \partial t^2 = -(A/t^2 + A^{-3})/4, \\ b(\partial^2 A / \partial t^2)(b; \lambda) = -\mu/4 - (\mu^3 4b^2)^{-1} \rightarrow d_2^* = -\mu/4,$$

and

$$\partial^3 A / \partial t^3 = (3/8) (A/t^3 + (tA^3)^{-1} + A^{-5}) + (8t^2 A)^{-1}.$$

Since  $A(tb_\lambda; \lambda)/b_\lambda$  is uniformly continuous at  $t = 1$ , (4.4)–(4.7) are satisfied with  $d_3^* = 3\mu/8$ . Rewriting  $T_\lambda$  into the form of Lai and Siegmund (1979), page 61, and applying Lemma 5 of Chow and Lai (1978), we have (2.35) with  $p = 1, 2$  under the condition that  $E|X|^{2p} < \infty$ .  $\square$

**EXAMPLE 2** [Woodroffe (1977) and Chow, Hsiung and Yu (1983)]. Let  $(X, Y), (X_1, Y_1), \dots$  be i.i.d. random vectors with  $EX = \mu > 0$  and  $EY = 0$ . Let  $\xi_n$  be a sequence of uniformly bounded random variables such that  $\xi_n$  is independent of  $(X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), \dots$ . Assume that  $a(t) > 0$ ,  $f(t)$  and  $h(t)$  are three functions such that  $t^{-\beta}a(t)$  is slowly varying for some  $\beta > 0$  and  $f(t)/t = o(1)$  as  $t \rightarrow \infty$  and that  $h(t)$  is bounded and continuous at  $t = 0$ . Define for each  $0 < \lambda < \infty$ ,

$$(5.12) \quad T = \inf\{n \geq n_\lambda: U_n < \lambda a(n)\},$$

where  $U_n = \bar{X}_n + h(\bar{Y}_n \xi_n) \bar{Y}_n^2 + f(n)/n$  and  $n_\lambda = K\lambda^{-\gamma}$  for some  $0 < \gamma < 1/\beta$ .

Suppose that  $E(X^+)^p < \infty$ ,  $E|Y|^p < \infty$  and  $\lim t^{-\beta}a(t) = 1$  as  $t \rightarrow \infty$ . Chow and Yu (1981) and Chow, Hsiung and Yu (1983) showed that

$$(5.13) \quad E|\lambda^{1/\beta} T_\lambda - \mu^{1/\beta}|^p = o(1), \quad \text{as } \lambda \rightarrow 0.$$

Let us apply our theorems. We shall first rewrite the stopping time as

$$(5.14) \quad T_\lambda = \inf\{n: Z_n > A(n; \lambda)\},$$

where  $Z_n = 2n\mu - nU_n$  and  $A(t; \lambda) = 2t\mu - \lambda ta(t)$ . Assume there exist two

functions  $u(t)$  and  $v(t)$  such that

$$(5.15) \quad t^{1/2}(d/dt)(f(t) - u(t)) = o(1) \quad \text{and} \quad u(t) = o(1), \quad \text{as } t \rightarrow \infty,$$

$$(5.16) \quad a(t) = t^\beta e^{v(t)}, \quad tv'(t) = o(1) \quad \text{and} \quad t^2 v''(t) = o(1), \quad \text{as } t \rightarrow \infty.$$

Then

$$(5.17) \quad \mu/\lambda = a(b_\lambda) = b_\lambda^\beta \exp[v(b_\lambda)],$$

$$(5.18) \quad d_\lambda = \mu(1 - \beta - b_\lambda v'(b_\lambda)) = \mu(1 - \beta + o(1)),$$

the conditions (2.36), (2.37), (3.3) and (3.4) are satisfied, and  $nh(\bar{Y}_n \xi_n) \bar{Y}_n^2$  is regular with  $\alpha = p = 1$  if  $EY^2 < \infty$  by Proposition 1. Assume, in addition, that  $E|X|^{p^*} + E|Y|^{p^*} < \infty$  for some  $p^* > 1 + 1/(\gamma\beta)$ . Then (2.35) holds with  $p = 1$ . Therefore if  $X - \mu(1 + \beta)$  does not have an arithmetic distribution, then all conditions in Theorems 1–3 hold with  $p = 1$ . We have

$$(5.19) \quad \begin{aligned} ET_\lambda &= b_\lambda + [\mu(\beta + b_\lambda v'(b_\lambda))]^{-1} f(b_\lambda) - (\mu\beta)^{-2} (\beta + 1) \text{Var}(X)/2 \\ &\quad + (\mu\beta)^{-1} EY^2 h(0) + (\mu\beta)^{-1} Es_\tau^2 / (2Es_\tau) + o(1), \end{aligned}$$

where  $s_n = \mu(1 + \beta)n - (X_1 + \cdots + X_n)$  and  $\tau = \inf\{n: s_n > 0\}$ .

**PROOF.** We only need to check (2.35) since other statements in the example are obvious. It follows from Lemma 5 of Chow and Lai (1978) that for some  $\theta > 0$ ,

$$\begin{aligned} P\{T_\lambda \leq \delta b_\lambda\} &\leq \sum_{k=0}^{\infty} P\{\max(S'_j: j \leq 2^{k+1}n_\lambda) > \theta 2^k n_\lambda\} \\ &\quad + \sum_{k=0}^{\infty} P\{\max(|S''_j|: j \leq 2^{k+1}n_\lambda) > \theta 2^k n_\lambda\} \\ &\leq \sum_{k=0}^{\infty} 2^{k+1} n_\lambda P\{\mu - X > \theta 2^k n_\lambda / (2j)\} + O(1) \sum_{k=0}^{\infty} [2^k n_\lambda]^{-j} \\ &\quad + \sum_{k=0}^{\infty} 2^{k+1} n_\lambda P\{|Y| > \theta 2^k n_\lambda / (2j)\} \\ &\leq 4(2j/\theta) E|X| I\{X > \theta n_\lambda / (2j)\} + O(1) n_\lambda^{-j} \\ &\quad + 4(2j/\theta) E|Y| I\{|Y| > \theta n_\lambda / (2j)\} \\ &= o(1) n_\lambda^{1-p^*} = o(1) b_\lambda^{-1}, \end{aligned}$$

where  $S'_n = n(\mu - \bar{X}_n)$ ,  $S''_n = n\bar{Y}_n$ ,  $j$  is an integer large enough and  $\delta < 1$ . Since

$$(\partial A / \partial t)(t; \lambda) = 2\mu - \lambda t a(t) [(1 + \beta)t^{-1} + v'(t)]$$

and

$$\begin{aligned} (\partial^2 A / \partial t^2)(t; \lambda) &= -\lambda t a(t) [(1 + \beta)t^{-1} + v'(t)]^2 \\ &\quad - \lambda t a(t) [-(1 + \beta)t^{-2} + v''(t)], \end{aligned}$$

the constants in (3.5) are given by (5.18) and  $d_1^* = \mu(1 - \beta)$  and  $d_2^* = -\mu\beta(1 + \beta)$ . Note that  $Z_n = S_n^* + \xi_n^*$  with  $S_n^* = 2\mu n - S_n$  and  $\xi_n^* = -n[h(\bar{Y}_n \xi_n) \bar{Y}_n^2 + f(n)/n]$ , we have (5.19).  $\square$

**EXAMPLE 3.**

$$(5.20) \quad T_\lambda = \inf\{n: S_n > \lambda n^\beta\}, \quad 0 \leq \beta < 1.$$

Then the conditions (2.10), (2.36), (2.37), (3.4) and (4.4)–(4.7) [all conditions in Theorems 1–4 on  $A(t; \lambda)$ ] are satisfied and the constants are given by

$$(5.21) \quad b = b_\lambda = (\lambda/\mu)^{1/(1-\beta)},$$

$$(5.22) \quad d = d_\lambda = \bar{d}_\lambda = d^* = d_1^* = \beta\mu,$$

$$(5.23) \quad d_2^* = \beta(\beta - 1)\mu \quad \text{and} \quad d_3^* = \beta(\beta - 1)(\beta - 2)\mu.$$

**PROOF.** Clearly, (2.1) implies that  $\mu b_\lambda = \lambda b_\lambda^\beta$ , which implies (5.21). By (5.20)

$$(5.24) \quad (\partial A / \partial t)(t; \lambda) - \beta\mu = \beta\mu((t/b_\lambda)^{\beta-1} - 1),$$

$$(5.25) \quad b_\lambda(\partial^2 A / \partial t^2)(t; \lambda) - \beta(\beta - 1)\mu = \beta(\beta - 1)\mu[(t/b_\lambda)^{\beta-2} - 1],$$

$$(5.26) \quad \begin{aligned} & b_\lambda^2(\partial^3 A / \partial t^3)(t; \lambda) - \beta(\beta - 1)(\beta - 2)\mu \\ &= \beta(\beta - 1)(\beta - 2)\mu[(t/b_\lambda)^{\beta-3} - 1]. \end{aligned}$$

The proof is complete.  $\square$

**6. Proof of Theorem 2.** We assume that  $EX = \mu = 1$  in this section without loss of generality. The following lemma states some useful results on random walk and renewal theory, which may be found in Chow (1973), Chow and Lai (1978) and Chow, Hsiung and Lai (1979).

**LEMMA 1.** Let  $S_n = X_1 + \cdots + X_n$ ,  $n \geq 1$ , and  $\tau(c, u)$  be defined by (1.2), and let  $p$  and  $\alpha$  be two constants with  $p \geq 1$  and  $1/2 < \alpha \leq 1$ .

(i) If  $E|X|^{(p+1)/\alpha} < \infty$ , then

$$\sum_{n=1}^{\infty} n^{p-1} P\left\{\max_{j \leq n} |S_j - j| \geq \theta n^\alpha\right\} < \infty, \quad \text{any } \theta > 0,$$

and

$$\sum_{n=1}^{\infty} n^{p-1} P\left\{\sup_{j \geq n} j^{-\alpha} |S_j - j| \geq \theta\right\} < \infty, \quad \text{any } \theta > 0.$$

(ii) If  $c^{(p+1)/\alpha} P\{|X| \geq c\} \rightarrow 0$  as  $c \rightarrow \infty$  and  $EX^2 < \infty$ , then

$$n^p P\left\{\max_{j \leq n} |S_j - j| \geq \theta n^\alpha\right\} \rightarrow 0, \quad \text{as } n \rightarrow \infty \text{ for any } \theta > 0,$$

and for any  $\theta > 0$  and  $K > 0$ ,

$$\lim_{c \rightarrow \infty} \sup_{K^{-1} \leq 1-u \leq K} c^p P\{\tau(c, u) \leq (1-u)^{-1}c - \theta c^\alpha\} = 0.$$

(iii) If  $E|X|^{2p} < \infty$ , then for any  $K > 0$ ,

$$\left\{ \left( \left( \tau(c, u) - (1-u)^{-1}c \right)^2 / c \right)^p; c \geq 1, K^{-1} \leq 1-u \leq K \right\} \text{ is u.i.}$$

LEMMA 2. Suppose that  $\xi$  is regular with  $p \geq 1$  and  $1/2 < \alpha \leq 1$  and that conditions (2.35) and (2.36) hold. If  $E|X|^{(p+1)/\alpha} < \infty$ , then

$$\lim_{b \rightarrow \infty} b^p P\{T \leq b - \theta b^\alpha\} = 0, \quad \text{for any } \theta > 0.$$

PROOF. On the event  $\{L < \delta b < n \leq b - \theta b^\alpha\}$ , by (2.36)

$$A(n; \lambda) \geq b + \mu^*(n - b) \geq n + (1 - \mu^*)\theta b^\alpha.$$

By (1.1) and (2.9)

$$\begin{aligned} & b^p P\{\delta b < T \leq b - \theta b^\alpha, L < \delta b\} \\ & \leq b^p P\{S_n + V_n + f(n) > n + (1 - \mu^*)\theta b^\alpha, \text{ some } \delta b < n \leq b\}. \end{aligned}$$

And by (2.30)  $f(n) = O(n^{1/2})$  and  $\{\max_{\delta b \leq n \leq b} |f(n)| \geq \theta' b^\alpha\} = \emptyset$  for large  $b$  and any given  $\theta' > 0$ . It follows from (2.35), (2.36), Lemma 1(ii) and (2.32) that

$$\begin{aligned} & b^p P\{T \leq b - \theta b^\alpha\} \\ & \leq b^p P\{T \leq \delta b\} + b^p P\{L \geq \delta b\} + b^p P\left\{\max_{n \leq b} |S_n - n| \geq \theta' b^\alpha\right\} \\ & \quad + b^p P\left\{\max_{\delta b \leq n \leq b} V_n \geq \theta' b^\alpha\right\} + b^p P\left\{\max_{\delta b \leq n \leq b} f(n) \geq \theta' b^\alpha\right\} \\ & = o(1), \end{aligned}$$

where  $\theta' = (1 - \mu^*)\theta/3 > 0$ .  $\square$

LEMMA 3. Suppose that  $\xi$  is regular with  $p \geq 1$  and  $1/2 < \alpha \leq 1$  and that condition (2.36) holds. If  $E|X|^{(p+1)/\alpha} < \infty$ , then there exists a constant  $K > 0$  such that

$$\lim_{b \rightarrow \infty} \sum_{n=n^*}^{\infty} n^{p-1} P\{T > n\} = 0,$$

where  $n^* = [b + Kb^\alpha]$ .

PROOF. By (2.33) there exists a  $w > 0$  such that

$$\sum_{n=1}^{\infty} n^{p-1} P\{V_n^- \geq wn^\alpha\} < \infty \quad \text{and} \quad w < 1 - \bar{d} \text{ if } \alpha = 1.$$

Let  $K$  be large enough and  $\theta > 0$  be small enough such that for large  $b$ ,

$$\begin{aligned} & (1 - \bar{d})(n - b) > (\theta + w)n^\alpha, \quad \text{for any } n \geq b + Kb^\alpha, \\ (6.1) \quad & A(n; \lambda) \leq b + \bar{d}(n - b) = n - (1 - \bar{d})(n - b) \\ & < n - (\theta + w)n^\alpha, \quad \text{any } n \geq n^*. \end{aligned}$$

By (2.29), Lemma 1(i), (2.33) and (2.30)

$$\begin{aligned}
 & P\{T > n, L < n\} \\
 & \leq P\{S_n + V_n + f(n) < n - (\theta + w)n^\alpha\}, \quad \text{any } n \geq n^*, \\
 & \sum_{n=n^*}^{\infty} n^{p-1} P\{T > n\} \\
 (6.2) \quad & \leq \sum_{n^*}^{\infty} n^{p-1} P\{L > n\} + \sum_{n^*}^{\infty} n^{p-1} P\{n - S_n > \theta n^\alpha/2\} \\
 & \quad + \sum_{n^*}^{\infty} n^{p-1} P\{V_n^- > wn^\alpha\} + \sum_{n^*}^{\infty} n^{p-1} P\{-f(n) > \theta n^\alpha/2\} \\
 & = o(1). \quad \square
 \end{aligned}$$

LEMMA 4. Under the conditions of Theorem 2(i),

$$\{((T - \tau)^+)^p; \lambda \in \Lambda\} \text{ is u.i.}$$

PROOF. Let  $n_1 = [b - \theta b^\alpha]$ ,  $n^* = [b + Kb^\alpha]$ ,  $T' = \max(n_1, \min(T, n^*))$  and  $\tau' = \max(n_1, \min(\tau, n^*))$ . For  $n > b + \theta b^\alpha$  we have

$$\begin{aligned}
 P\{\tau > n\} & \leq P\{S_n - dn \leq b(1-d) - f(b)\} \\
 & \leq P\{n - S_n \geq (n-b)(1-d) - f(b)\}.
 \end{aligned}$$

It follows from Lemma 1(i) and (2.30) that for  $n_2 = [b + \theta b^\alpha]$ ,

$$(6.3) \quad \lim_{b \rightarrow \infty} \sum_{n=n_2}^{\infty} n^{p-1} P\{\tau > n\} = 0, \quad \text{for any } \theta > 0.$$

And by Lemmas 1(ii), 2 and 3

$$(6.4) \quad \lim_b E|T - T'|^p = \lim_b E|\tau - \tau'|^p = 0$$

and

$$\begin{aligned}
 P\{T' > \tau' + n\} & \leq P\{L \geq n_1\} + P\{\tau \leq n_1\} \\
 & \quad + P\{L < n_1 < \tau \leq \tau + n < T' \leq n^*\}.
 \end{aligned}$$

Making use of (2.29), (2.30) and Lemma 1(ii), we find that

$$\begin{aligned}
 & \sum_{n_0}^{\infty} n^{p-1} P\{T' > \tau' + n\} \\
 (6.5) \quad & = \sum_{n_0}^{n^*} n^{p-1} P\{T' > \tau' + n\} \\
 & = o(1) + \sum_{n_0}^{n^*} n^{p-1} P\{S_{\tau+n} + V_{\tau+n} + f(\tau+n) \\
 & \quad \leq A(\tau+n; \lambda), \tau \in (n_1, n^* - n)\}.
 \end{aligned}$$

On the event  $\{n_1 < \tau \leq \tau + n < n^*\}$ , we have by (2.36), (2.34) and (2.30)

$$A(\tau + n; \lambda) \leq \mu^* n + (A(\tau; \lambda) - b - d\tau) + S_\tau + f(b),$$

and by (2.37) there exists a constant  $K^* < \infty$  such that

$$(6.6) \quad \begin{aligned} |A(m; \lambda) - b - dm| &\leq K^*(m - b)^2/b, \quad n_1 \leq m \leq n^*, \\ |f(\tau + n) - f(b)| &\leq K^*|\tau + n - b|n_1^{-1/2} \\ &\leq n(1 - \mu^*)/5 + K^*|\tau - b|n_1^{-1/2}, \end{aligned}$$

and

$$\begin{aligned} S_{\tau+n} + V_{\tau+n} + f(\tau + n), \quad \text{on } \{T' > \tau' + n\}, \\ \leq \mu^* n + (A(\tau; \lambda) - b - d\tau) + S_\tau + f(b) \\ \leq \mu^* n + K^*(\tau - b)^2/b + S_\tau + K^*|\tau - b|n_1^{-1/2} + f(\tau + n) + n(1 - \mu^*)/5. \end{aligned}$$

Therefore, letting  $\theta' = (1 - \mu^*)/5$ , we have by (6.5)

$$\begin{aligned} \sum_{n_0}^{\infty} n^{p-1} P\{T' > \tau' + n\} &\leq \sum_{n_0}^{\infty} n^{p-1} P\{S_\tau + n - S_{\tau+n} \geq \theta' n\} \\ &\quad + \sum_{n_0}^{\infty} n^{p-1} P\left\{\max_{n_1 \leq j \leq n^*} V_j^- > \theta' n\right\} \\ &\quad + \sum_{n_0}^{\infty} n^{p-1} P\{K^*(\tau - b)^2/b > \theta' n\} \\ &\quad + \sum_{n_0}^{\infty} n^{p-1} P\{K^*|\tau - b|n_1^{-1/2} > \theta' n\} + o(1). \end{aligned}$$

It follows from Lemma 1(i), (2.31), Lemma 1(iii) and the condition  $E|X|^{2p} < \infty$  that

$$\sum_{n_0}^{\infty} n^{p-1} P\{T > \tau' + n\} \rightarrow 0, \quad \text{as } \min(n_0, b) \rightarrow \infty.$$

This proves the uniform integrability of  $\{(T - \tau)^{+p}\}$  since the uniform integrability of  $\{T^p; b \leq b^*, \lambda \in \Lambda\}$  for any given  $b^*$  is implied by (6.2) in the proof of Lemma 3.  $\square$

**LEMMA 5.** *Let  $Y$  be an integer-valued random variable with finite  $p$ th moment,  $p \geq 1$ . Suppose that there exist constants  $c$ ,  $\theta$  and  $\delta^*$  such that  $0 \leq \delta^* < 1$ ,  $\theta(4\delta^*)^p \leq 2$  and for some  $m > 1/(1 - \delta^*)$ ,*

$$\sum_{n=m}^{\infty} n^{p-1} P\{Y \leq n\} \leq c + \theta \sum_{n=m}^{\infty} n^{p-1} P\{\delta^* Y \geq n\} < \infty.$$

Then

$$\sum_{n=m}^{\infty} n^{p-1} P\{Y \geq n\} \leq 2c.$$

PROOF. Clearly,

$$\begin{aligned} \sum_m n^{p-1} P\{\delta^* Y \geq n\} &\leq 2^{p-1} \int_{m-1}^{\infty} y^{p-1} P\{\delta^* Y \geq y\} dy \\ &\leq 2^{p-1} (\delta^*)^p \int_{(m-1)/\delta^*}^{\infty} y^{p-1} P\{Y \geq y\} dy \\ &\leq 4^{p-1} (\delta^*)^p \sum_m n^{p-1} P\{Y \geq n\} \leq (4\theta)^{-1} 2 \sum_m n^{p-1} P\{Y \geq n\}. \end{aligned}$$

□

PROOF OF THEOREM 2. (i) Let  $n_1 = [b - \theta b^\alpha]$ ,  $n_2 = [b + \theta b^\alpha]$ ,  $T' = \max(n_1, \min(T, n_2))$  and  $\tau' = \max(n_1, \min(\tau, n_2))$ . Though  $n_2$  is different from  $n^*$  in Lemma 3,  $(T - T')^+ \leq (T - \tau)^+ + (\tau - n_2)^+$ , and by Lemma 4, (6.3) and (6.4) we have

$$(6.7) \quad \lim_{b \rightarrow \infty} E|T - T'|^p = \lim_{b \rightarrow \infty} E|\tau - \tau'|^p = 0.$$

Clearly,

$$\begin{aligned} P\{\tau' > T' + n\} &\leq P\{L \geq n_1\} + P\{T \leq n_1\} + P\{\tau \geq n_2\} \\ &\quad + P\{L < n_1 < T \leq T' + n < \tau < n_2\}. \end{aligned}$$

By (2.29), Lemma 2 and (6.3)

$$\begin{aligned} (6.8) \quad \sum_{n=n_0}^{\infty} n^{p-1} P\{\tau' > T' + n\} &= \sum_{n=n_0}^{n_2-n_1} n^{p-1} P\{\tau' > T' + n\} \\ &\leq \sum_{n_0}^{n_2-n_1} n^{p-1} P\{L < n_1 < T \leq T' + n < \tau < n_2\} + o(1). \end{aligned}$$

On the event  $\{L < n_1 < T \leq T' + n < \tau < n_2\}$ ,

$$(6.9) \quad \begin{aligned} S_{T'+n} + f(b) &\leq b + d(T' + n - b) \\ &\leq \mu^* n + (b + d(T - b) - A(T; \lambda)) + S_T + V_T + f(T), \end{aligned}$$

and there exists some finite constant  $K^*$  that does not depend on  $\theta$  such that

$$\begin{aligned} (6.10) \quad f(T) - f(\tau) &\leq |\tau - T| K^* n_1^{-1/2}, \quad \text{by (2.30),} \\ f(\tau) - f(b) &\leq K^* |\tau - b| b^{-1/2}, \quad \text{by (2.30),} \\ (b + d(T - b) - A(T; \lambda)) - (b + d(\tau - b) - A(\tau; \lambda)) \\ &\leq |\tau - T| (\theta K^* b^{\alpha-1}), \quad \text{for small } \theta \text{ by (2.37),} \\ (b + d(\tau - b) - A(\tau; \lambda)) &\leq K^* (\tau - b)^2 / b, \quad \text{by (6.6),} \\ (b + d(T - b) - A(T; \lambda)) + f(T) - f(b) \\ &\leq \delta^* |\tau' - T'| + K^* (\tau - b)^2 / b + K^* |\tau - b| b^{-1/2}, \end{aligned}$$

where  $\delta^* = \theta K^* b^{\alpha-1} + K^* b^{-1/2}$ , and by (6.9)

$$(6.11) \quad \begin{aligned} S_{T+n} - S_T &\leq \mu^* n + V_T + \delta^* |\tau' - T'| \\ &\quad + K^* (\tau - b)^2/b + K^* |\tau - b| b^{-1/2}. \end{aligned}$$

Therefore it follows from (6.8) and (6.11) that for  $\theta' = (1 - \mu^*)/5$ ,

$$(6.12) \quad \begin{aligned} &\sum_{n_0}^{\infty} n^{p-1} P\{\tau' > T' + n\} \\ &\leq \sum_{n_0}^{\infty} n^{p-1} P\{S_T + n - S_{T+n} \geq \theta' n\} \\ &\quad + \sum_{n_0}^{\infty} n^{p-1} P\left\{\max_{n_1 \leq j \leq n_2} V_j \geq \theta' n\right\} + \sum_{n_0}^{\infty} n^{p-1} P\{\delta^*(T - \tau) \geq \theta' n\} \\ &\quad + \sum_{n_0}^{\infty} n^{p-1} P\{K^*(\tau - b)^2/b + K^* |\tau - b| n^{-1/2} \geq \theta' n\} \\ &\quad + \sum_{n_0}^{\infty} n^{p-1} P\{\delta^*(\tau' - T') \geq \theta' n\} + o(1), \quad \text{as } \min(n_0, b) \rightarrow \infty. \end{aligned}$$

Since  $\theta$  is arbitrary, we can choose  $\theta$  small enough such that  $(4\delta^*/\theta')^p \leq 2$  for the  $\delta^*$  specified in (6.10). Hence it follows from (6.12), Lemma 1(i), (2.31), Lemmas 4 and 1(iii) that

$$\sum_{n_0}^{\infty} n^{p-1} P\{\tau' - T' > n\} \leq \sum_{n_0}^{\infty} n^{p-1} P\{\delta^*(\tau' - T') \geq \theta' n\} + o(1),$$

as  $\min(n_0, b) \rightarrow \infty$ .

And by (6.7) and Lemmas 4 and 5

$$\sum_{n_0}^{\infty} n^{p-1} P\{\tau' - T' > n\} = o(1)$$

and

$$\{|T - \tau|^p; \lambda \in \Lambda\} \text{ is u.i.}$$

(ii) For the case where  $\partial^2 A / \partial t^2 = 0$ , the term  $(\tau - b)^2/b$  disappears throughout the proof of (i).

(iii) Let  $n^* = [b + Kb^\alpha]$ . By (6.1)

$$P\{N > n, L < n\} \leq P\left\{\sup_{j \geq n} j - S_j - (\theta + w)j^\alpha + V_j^- - f(j) \geq 0\right\},$$

any  $n \geq n^*$ .

The term  $\sup_{j \geq n} V_{\tau+j}^-/j$  is controlled on  $n_1 \leq \tau \leq n^*$  by

$$\begin{aligned} \sup_{j \geq n} V_{\tau+j}^-/j &\leq \max_{n \leq j \leq K^* b^\alpha} V_{\tau+j}^-/j + \sup_{j \geq \max(n, K^* b^\alpha)} V_{\tau+j}^-/j \\ &\leq \max_{n \leq j \leq n^* + K^* b^\alpha} V_j^-/n + \sup_{j \geq n} V_j^-/j^\alpha \sup_{j \geq K^* b^\alpha} (\tau + j)^\alpha/j. \end{aligned}$$

Since  $K^*$  is arbitrary, we can take  $K^*$  large enough such that

$$\sup_{j \geq K^* b^*} (n^* + j)^\alpha / j \leq 1 + \delta^*, \quad \alpha = 1, \\ \leq \delta^*, \quad \alpha < 1,$$

for any given  $\delta^* > 0$ .

The rest of the proof is similar to the proof of (i) and is omitted.  $\square$

**7. Proof of Theorem 4.** We shall keep all the notation and definitions introduced in Sections 1–4 and assume  $Z_n = S_n$  so that the stopping time  $T$  is defined by (4.1).

Let  $n_1 = [b_\lambda - \theta b_\lambda]$  and  $n_2 = [b_\lambda + \theta b_\lambda]$  for some small  $\theta > 0$ , where  $[y]$  is the largest integer in  $(-\infty, y]$ . We shall use the notation

$$(7.1) \quad T' = T'_\lambda = \inf\{n \geq n_1: S_n > A(n; \lambda) \text{ or } n = n_2\},$$

$$(7.2) \quad R' = R'_\lambda = S_{T'} - A(T'; \lambda),$$

$$(7.3) \quad g(t; \lambda) = A(t; \lambda) - (\mu b_\lambda + d_\lambda(t - b_\lambda)).$$

**LEMMA 6.** Suppose that (4.3) and (4.4) hold. If  $EX^4 < \infty$ , then

$$(7.4) \quad \text{Var}(T_\lambda) - \text{Var}(T'_\lambda) = o(1), \quad \text{as } b_\lambda \rightarrow \infty,$$

$$(7.5) \quad \text{Cov}(T_\lambda, R_\lambda) - \text{Cov}(T'_\lambda, R'_\lambda) = o(1), \quad \text{as } b_\lambda \rightarrow \infty \text{ if (4.12) holds,}$$

and

$$E(T_\lambda - b_\lambda)^3 / b_\lambda - E(T'_\lambda - b_\lambda)^3 / b_\lambda = o(1), \quad \text{as } b_\lambda \rightarrow \infty.$$

Since by (4.12)  $RI\{T > n_2\} \leq |X_T| + KT^2$  for some  $K < \infty$  and

$$ET|X_T|I\{T > n_2\} \leq E|X|E(T^2 - n_2^2)^+,$$

the proof of Lemma 6 is similar to the proofs of Lemmas 2 and 3 and is therefore omitted.

**LEMMA 7.** Suppose that the conditions (4.3)–(4.6) hold and that  $X$  has a strongly nonlattice distribution with  $EX^4 < \infty$ . Then

$$(7.6) \quad \text{Var}(R'_\lambda + g(T'_\lambda)) = - \int_0^\infty x^2 dG(x, d_1^*) - r^2(d_1^*) \\ + (\sigma/\mu_0)^4 (d_2^*)^2 / 2 + o(1), \quad \text{as } b \rightarrow \infty,$$

where  $\mu_0 = \mu - d_1^*$ .

**PROOF.** Let  $\tau = \tau_\lambda = \tau(c_\lambda, d_\lambda)$ ,  $\tau(c, u)$  be defined by (1.2) and  $c_\lambda = b_\lambda(\mu - d_\lambda)$ . By the definitions

$$(7.7) \quad S_{T'} - S_\tau - d(T' - \tau) + R(c_\lambda, d_\lambda) = R'_\lambda + g(T'_\lambda), \\ P\{R(c, u) > x\} \leq \int_x^\infty E\tau(y - x, u) dP\{X - u \leq y\}.$$

Since

$$(7.8) \quad \sup\{E\tau(c, u)/c: u \leq \mu^*, c \geq 1\} < \infty, \\ \{(R(c_\lambda, d_\lambda))^3, \lambda \in \Lambda\} \text{ is u.i.}$$

By Theorem 2 and Lemma 6

$$\{(T'_\lambda - \tau_\lambda)^2, \lambda \in \Lambda\} \text{ is u.i.}$$

It follows that

$$(7.9) \quad \{(S_{T'} - S_\tau)^4, \lambda \in \Lambda\} \text{ is u.i.,} \\ \{(R'_\lambda + g(T'_\lambda))^2, \lambda \in \Lambda\} \text{ is u.i., by (7.7).}$$

Taking a Taylor expansion, we have

$$g(T'_\lambda) = d_2^*(T'_\lambda - b)^2 b^{-1}(1 + o(1))/2.$$

It follows from Theorem 1 that

$$P\{R'_\lambda > x, g(T'_\lambda) > t(d_2^*/2)(\sigma/\mu_0)^2\} = G(x, d_1^*)P\{(N(0, 1))^2 > t\} + o(1).$$

This and (7.9) finish the proof.  $\square$

**LEMMA 8.** *Under the conditions of Theorem 4, the following relation holds:*

$$(7.10) \quad \text{Cov}(T'_\lambda, R'_\lambda) = C_3^*/\mu_0 - (\sigma/\mu_0)^2 d_2^* \left[ \left(1 - (\sigma/\mu_0)^2\right)/2 \right. \\ \left. + \sum_{n=1}^{\infty} P\{S_n - nd_1^* \leq 0\} \right] + o(1),$$

as  $b \rightarrow \infty$ , where  $\mu_0 = \mu - d_1^*$  and

$$C_3^* = \int_0^\infty [ER(x, d_1^*) - r(d_1^*)] P\left\{\max_n (S_n - nd_1^*)^- \leq x\right\} dx.$$

**PROOF.** For  $j = 0, \dots, 4$ ,  $i = 1, 2$ , there is a finite  $K$  such that

$$b^{j-2}|S_{n_i}|^{4-j} \leq Kb^2 + (S_{n_i} - \mu n_i)^4 Kb^{-2} I\{|S_{n_i} - \mu n_i| \geq b\}.$$

It follows from (4.8) and Chow and Lai (1978) that

$$(7.11) \quad b^{j-2} E|S_{n_i}|^{4-j} I\{T' = n_i\} \rightarrow 0, \text{ for } i = 1, 2 \text{ and } 0 \leq j \leq 4.$$

Let  $d(t; \lambda) = A(t; \lambda) - A(t-1; \lambda)$  and

$$(7.12) \quad F(x; n) = F(x; n, \lambda) = P\{T'_\lambda \geq n, A(n-1; \lambda) - S_{n-1} \leq x\}.$$

Then we have

$$(7.13) \quad P\{R'_\lambda > x, T'_\lambda = n\} \\ = \int_0^\infty P\{R(y, d(n; \lambda)) > x\} d[F(y; n, \lambda) - F(y; n+1, \lambda)],$$

for  $n_1 < n < n_2$ .

By (7.11) and (7.13)

$$(7.14) \quad \begin{aligned} & ER'_\lambda T'_\lambda \\ &= \sum_{n=n_1+1}^{n_2-1} \int_0^\infty n ER(y, d(n; \lambda)) d[F(y; n) - F(y; n+1)] + o(1), \end{aligned}$$

and similarly,

$$(7.15) \quad \begin{aligned} & Er(d(T'_\lambda; \lambda)) T'_\lambda \\ &= \sum_{n=n_1+1}^{n_2-1} \int_0^\infty nr(d(n; \lambda)) d[F(y; n) - F(y; n+1)] + o(1). \end{aligned}$$

It follows from Theorems 5 and 6 and (4.8) that there exists a decreasing function  $h(x) > 0$  such that for any  $u$  with  $-\infty < u^* \leq u \leq \mu^* < \mu$ ,

$$(7.16) \quad |Er(x, u) - r(u)| \leq h(x) \quad \text{and} \quad \int_0^\infty h(x) dx < \infty.$$

By (4.4)–(4.6) we can choose  $u^*$  such that

$$(7.17) \quad u^* \leq d(n; \lambda) \leq \mu^*, \quad \text{for all } |n - b_\lambda| \leq \theta b \quad \text{and } b \text{ large.}$$

Let  $n$  be an integer with  $\theta b/2 \leq b - n \leq \theta b$ . By (4.4)

$$(7.18) \quad A(n; \lambda) - \mu n \geq (\mu - \mu^*)(b - n) \geq (\mu - \mu^*)\theta b/2.$$

For  $u^* \leq u \leq \mu^*$ ,

$$\begin{aligned} & b \left| \int_0^\infty (ER(x, u) - r(u)) dF(x; n+1) \right| \\ & \leq b \int_0^\infty h(x) dP\{A(n; \lambda) - S_n \leq x\} \quad [\text{by (7.16)}] \\ & \leq bh(b\theta') + bh(0)P\{A(n; \lambda) - S_n \leq \theta'b\} \\ & \leq bh(\theta'b) + bh(0)P\{S_n - \mu n \geq (\mu - \mu^*)\theta b/2 - \theta'b\} \quad [\text{by (7.18)}] \\ & \leq o(1) + bh(0)P\{S_n - \mu n \geq \theta'b\} \\ & = o(1), \quad \text{by taking } \theta' \leq (\mu - \mu^*)\theta/4. \end{aligned}$$

Also, for  $\theta b/2 \leq n - b \leq \theta b$ ,

$$(7.19) \quad A(n; \lambda) \leq \mu b + (n - b)\mu^*, \quad \text{by (4.4),}$$

and

$$P\{T' > n\} \leq P\{n\mu - S_n \leq (n - b)(\mu - \mu^*)\} = o(b^{-1}).$$

Therefore, for  $\theta b/2 \leq |n - b| \leq \theta b$ ,  $u^* \leq u \leq \mu^*$ , we have

$$(7.20) \quad b \left| \int_0^\infty (ER(x, u) - r(u)) dF(x; n) \right| = o(1), \quad \text{by (7.16) and (7.17).}$$

It follows from (7.14), (7.15) and (7.20) that

$$\begin{aligned}
 & \text{Cov}(R'_\lambda, T'_\lambda) - \text{Cov}(r(d(T'_\lambda; \lambda)), T'_\lambda) \\
 &= \text{Cov}([R'_\lambda - r(d(T'_\lambda; \lambda))], T'_\lambda) \\
 &= \sum_{n=n_1+1}^{n_2-1} \int_0^\infty (n - ET'_\lambda) [ER(x, d(n; \lambda)) - r(d(n; \lambda))] \\
 &\quad \times d[F(x; n) - F(x; n+1)] + o(1) \\
 (7.21) \quad &= \sum_{n=n_1+1}^{n_2-1} \int_0^\infty ER(x, d(n; \lambda)) - r(d(n; \lambda)) dF(x; n) \\
 &\quad + \sum_{n=n_1+1}^{n_2-1} \int_0^\infty (n - ET' - 1) \delta^*(y; n, \lambda) dF(y; n) + o(1) \\
 &= I_1 + I_2 + o(1), \quad \text{say,}
 \end{aligned}$$

where

$$\delta^*(x; n, \lambda) = \int_{d(n-1; \lambda)}^{d(n; \lambda)} (\partial/\partial u) [ER(x, u) - r(u)] du.$$

By Theorem 6 there exists a function  $h^*(x)$  such that

$$(7.22) \quad |(\partial/\partial u) [ER(x, u) - r(u)]| \leq h^*(x) \quad \text{for } u^* \leq u \leq \mu^*,$$

and

$$(7.23) \quad \sum_{n=0}^{\infty} \sup_{n \leq x < n+1} h^*(x) \leq K < \infty.$$

By (7.17) and the definitions

$$|\delta^*(x; n, \lambda)| \leq h^*(x) |d(n; \lambda) - d(n-1; \lambda)|, \quad \text{for large } b.$$

And by (4.5) and (4.6), we can bound the second term in (7.21) as

$$\begin{aligned}
 (7.24) \quad |I_2| &\leq \sum_{n=n_1+1}^{n_2-1} \int_0^\infty |n - ET'_\lambda - 1| h^*(x) K b^{-1} dF(x; n, \lambda) \\
 &= \sum_{n=n_1+1}^{n_2-1} \int_{\theta''b}^\infty + \sum_{|n-b| \leq \theta'b} \int_0^{\theta''b} + \sum_{\theta'b < |n-b| \leq \theta b} \int_0^{\theta''b}.
 \end{aligned}$$

We shall prove that the three terms given previously tend to zero as  $b_\lambda$  tends to infinity by choosing suitable  $\theta'$  and  $\theta''$ . Clearly,

$$P\{A(n; \lambda) - S_n \geq \theta''b\} \leq P\{\mu n - S_n \geq \mu n - A(n; \lambda) + \theta''b\}.$$

For  $|n - b| \leq K\theta b$  we have by (4.5) and (4.6)

$$\begin{aligned}
 (7.25) \quad |A(n; \lambda) - (\mu b + d(n - b))| &= |g(n; \lambda)| \\
 &\leq K^*(n - b)^2/b, \quad \text{for some } K^* < \infty,
 \end{aligned}$$

and

$$n\mu - A(n; \lambda) \geq -(\mu - d)|n - b| - K^*(n - b)^2/b \geq -\theta''b/2,$$

provided that  $|n - b| \leq \theta b$  and that

$$(7.26) \quad 0 < \theta(\mu - d + K^*\theta) \leq \theta''/2.$$

Hence

$$\begin{aligned} P\{A(n; \lambda) - S_n \geq \theta''b\} &\leq P\{\mu n - S_n \geq \theta''b/2\} \\ &= o(b^{-2}), \quad n_1 \leq n \leq n_2. \end{aligned}$$

And the first term on the right-hand side of (7.24) is  $o(1)$ . Let  $\lambda' = (\lambda, x)$  and  $A'(t; \lambda') = A(t; \lambda) - x$ . Then by (4.4)

$$0 \leq b - b' \leq x/(\mu - \mu^*), \quad \text{for } 0 \leq x \leq (\mu - \mu^*)(1 - \theta)b,$$

where  $b' = \sup\{t: A'(t; \lambda') \leq \mu t\}$ .

Therefore, if  $\theta''$  is small, the family  $\{A'(t; \lambda'), 0 \leq x \leq \theta''b, \lambda \in \Lambda\}$  still satisfies the conditions (4.3)–(4.7). It follows from Corollary 2 that

$$\begin{aligned} &\sum_{n=n_1+1}^{n_2-1} P\{A'(n; \lambda') < S_n \leq A'(n; \lambda') + 1\} \\ &= \sum_{n=n_1+1}^{n_2-1} P\{x - 1 \leq A(n; \lambda) - S_n < x\} \\ &\leq K < \infty, \quad \text{for any } x \leq \theta''b \quad \text{and } \lambda \in \Lambda. \end{aligned}$$

The second term on the right-hand side of (7.24) is bounded by

$$\begin{aligned} &(\theta'b + |ET' - 1 - b|)Kb^{-1}K \sum_{j=0}^{\infty} \sup_{j \leq x < j+1} h^*(x) \\ &\leq \theta'K^3 + o(1), \quad \text{for some } K \text{ not depending on } \theta'. \end{aligned}$$

It follows from (7.25) that

$$\begin{aligned} |A(n; \lambda) - \mu n| &\geq |(\mu - \mu^* - K^*\theta/b)(n - b)|, \\ P\{T > n, A(n; \lambda) - S_n \leq x\} &= F(x; n + 1) \\ &\leq P\{|S_n - \mu n| \geq (\mu - \mu^* - K^*\theta/b)\theta'b\} \\ &= o(b^{-1}), \end{aligned}$$

for  $\theta'b \leq |n - b| \leq \theta b$ .

The third term on the right-hand side of (7.24) is bounded by

$$K^2 \sum_{j=m}^{\infty} \sup_{j \leq x < j+1} h^*(x) + K^2(n_2 - n_1)o(b^{-1}) = o(1).$$

Note that the constant  $\theta$  is arbitrary when we define the stopping time  $T'$  in (7.1) and the choice of  $\theta''$  only depends on the parameters in the conditions (4.3)–(4.6), we can always find a  $\theta''$  such that (7.26) is satisfied. By letting  $b_\lambda \rightarrow \infty$  first and then  $\theta' \rightarrow 0$ , we have  $I_2 = o(1)$  and it follows from (7.21) that

$$\text{Cov}([R'_\lambda - r(d(T'_\lambda; \lambda))], T'_\lambda) = I_1 + o(1).$$

Since by (7.22) and (7.17)

$$\begin{aligned} &|[ER(x, d(n; \lambda)) - r(d(n; \lambda))] - [ER(x, d_1^*) - r(d_1^*)]| \\ &\leq h^*(x)|d(n; \lambda) - d_1^*|, \end{aligned}$$

we can repeat the argument again and have

$$\begin{aligned} & \text{Cov}([R' - r(d(T'; \lambda))], T') \\ &= \int_0^\infty [ER(x, d_1^*) - r(d_1^*)] d \left[ \sum_{n=n_1}^{n_2} F(x; n, \lambda) \right] + o(1). \end{aligned}$$

It follows from Corollary 3 that

$$\begin{aligned} & \text{Cov}([R' - r(d(T'; \lambda))], T')\mu_0 \\ (7.27) \quad &= \int_0^\infty [ER(x, d_1^*) - r(d_1^*)] P\left(\max_n (S_n - nd_1^*)^- \leq x\right) dx + o(1). \end{aligned}$$

Now let us consider  $\text{Cov}(r(d(T'_\lambda; \lambda)), T'_\lambda)$ . By Wiener–Hopf factorization [Feller (1966), page 605], the derivative of  $r(u)$  is continuous and

$$(\partial r / \partial u)(d_1^*) = -\left(1 - (\sigma/\mu_0)^2\right)/2 - \sum_{n=1}^\infty P\{S_n - nd_1^* \leq 0\}.$$

It follows from (4.5) and (4.6) that

$$r(d(T'_\lambda; \lambda)) - r(d_1^*) = [(\partial r / \partial u)(d_1^*)] d_2^*(T'_\lambda - b)(1 + o(1))/b.$$

Hence

$$\begin{aligned} & \text{Cov}(r(d(T'; \lambda)), T') \\ &= -d_2^*(\sigma/\mu_0)^2 \left[ \left(1 - (\sigma/\mu_0)^2\right)/2 + \sum_{n=1}^\infty P\{S_n - nd_1^* \leq 0\} \right]. \quad \square \end{aligned}$$

**LEMMA 9.** *Suppose that the conditions (4.4)–(4.7) hold and that  $X$  has a strongly nonlattice distribution with finite fourth moment. Let  $T'_\lambda$  be defined by (7.1) and  $\mu_0 = \mu - d_1^*$ . Then*

$$\begin{aligned} & \text{Cov}(T'_\lambda, g(T'_\lambda; \lambda)) \\ (7.28) \quad &= \mu_0^{-3} [d_2^* \sigma^2 r(d_1^*) - d_2^* E(X - \mu)^3 / 2] \\ &+ \mu_0^{-4} (d_3^* + 3d_2^*) \sigma^4 / 2 + \mu_0^{-5} 2(d_2^*)^2 \sigma^4 + o(1) \end{aligned}$$

and

$$\begin{aligned} E(T'_\lambda - b)^3 / b &= \mu_0^{-3} [3\sigma^2 r(d_1^*) - E(X - \mu)^3] \\ &+ \mu_0^{-4} (3\sigma^4) + \mu_0^{-5} (9d_2^* \sigma^4 / 2) + o(1). \end{aligned}$$

**PROOF.** By (4.5) and (4.6)

$$(7.29) \quad g(T'_\lambda; \lambda) = d_2^* b^{-1} (T'_\lambda - b)^2 / 2 + d_3^* b^{-2} (T'_\lambda - b)^3 (1/6 + o(1)),$$

where  $o(1)$  is uniformly bounded by a constant  $K$ . By (7.1)–(7.3)

$$(7.30) \quad S_{T'} - \mu T'_\lambda = (\mu - d)(b - T'_\lambda) + g(T'_\lambda; \lambda) + R'_\lambda,$$

$$(7.31) \quad |g(T'_\lambda; \lambda)/b| \leq K\theta^2 < \infty \quad \text{and} \quad T'/b \leq 2, \quad \text{a.s.}$$

It follows from Theorem 3 that

$$(7.32) \quad ET'_\lambda = b + \mu_0^{-1} \left( r(d_1^*) + (d_2^*(\sigma/\mu_0)^2/2) \right) + o(1).$$

Since

$$(7.33) \quad \begin{aligned} &P\{R'_\lambda > x, T'_\lambda \neq n_1, T'_\lambda \neq n_2\} \\ &\leq P\{X_{T'} > x\} \leq ET'P\{X > x\} \leq 2bP\{X > x\}, \\ &\quad \{(R')^3/b, \lambda \in \Lambda\} \text{ is u.i., by (7.11).} \end{aligned}$$

By Lemma 5 of Chow and Yu (1981)

$$(7.34) \quad \{(S_{T'} - \mu T'_\lambda)^4/b^2, (T'_\lambda - b)^4/b^2\} \text{ is u.i.}$$

And by Theorem 1 with  $(\sigma^*)^2 = (\sigma/\mu_0)^2$ ,

$$(7.35) \quad P\{R'_\lambda > x, T'_\lambda - b > t\sigma^*\sqrt{b}\} = G(x, d_1^*)P\{N(0,1) > t\} + o(1).$$

By (7.29), (7.34), (7.35) and (7.32)

$$(7.36) \quad \begin{aligned} &\text{Cov}(T', g(T'; \lambda)) \\ &= d_2^*E(T' - b)^3/(2b) + d_3^*(\sigma^*)^4E(N(0,1))^4/6 \\ &\quad - d_2^*E(T' - b)^2(2b)^{-1}(ET' - b) + o(1) \\ &= d_2^*E(T' - b)^3/(2b) + d_3^*(\sigma^*)^4/2 \\ &\quad - d_2^*(\sigma^*)^2[r(d_1^*) + d_2^*(\sigma^*)^2/2]/(2\mu_0) + o(1). \end{aligned}$$

By (7.29), (7.31) and (7.34)

$$\{(g(T'; \lambda))^2, |g(T'; \lambda)|^3/b\} \text{ is u.i.}$$

By (7.33) and (7.35)

$$E(g(T'; \lambda) + R')^3/b = o(1).$$

Since  $(T' - b)/b$  is bounded, by (7.9) and (7.35)

$$E(T' - b)(g(T'; \lambda) + R')^2/b = o(1).$$

Therefore by (7.30)

$$(7.37) \quad \begin{aligned} E(T' - b)^3/b &= [3\mu_0^2E(T' - b)^2(g(T'; \lambda) + R')/b \\ &\quad - E(S_{T'} - \mu T')^3/b]/\mu_0^3 + o(1). \end{aligned}$$

Since  $S_n - \mu n$ ,  $n \geq 1$ , are partial sums of i.i.d. mean zero random variables and  $T'$  is a bounded stopping time, it follows from Chow, Robbins and Teicher (1965) that

$$(7.38) \quad E(S_{T'} - \mu T')^3 = ET'E(X - \mu)^3 + 3\sigma^2ET'(S_{T'} - \mu T').$$

Again, by (7.34), (7.30) and (7.35)

$$(7.39) \quad \begin{aligned} ET'(S_{T'} - \mu T')/b &= -\mu_0E(T' - b)^2/b + o(1). \\ &= -\mu_0(\sigma^*)^2 + o(1). \end{aligned}$$

By (7.29), (7.34), (7.9) and (7.35)

$$(7.40) \quad E(T' - b)^2(g(T'; \lambda) + R')/b = 3d_2^*(\sigma^*)^4/2 + (\sigma^*)^2 r(d_1^*) + o(1).$$

It follows from (7.37)–(7.40) that

$$\begin{aligned} E(T' - b)^3/b &= 3[3d_2^*(\sigma^*)^4/2 + (\sigma^*)^2 r(d_1^*)]/\mu_0 \\ &\quad - E(X - \mu)^3/\mu_0^3 + 3\sigma^2(\sigma^*)^2/\mu_0^2 + o(1). \end{aligned}$$

Since  $\sigma^* = \sigma/(\mu - d_1^*) = \sigma/\mu_0$ , by algebra

$$\begin{aligned} E(T' - b)^3/b &= \mu_0^{-3}[3\sigma^2 r(d_1^*) - E(X - \mu)^3] + \mu_0^{-4}(3\sigma^4) \\ &\quad + \mu_0^{-5}(9d_2^*\sigma^4/2) + o(1). \end{aligned}$$

Hence by (7.36)

$$\begin{aligned} \text{Cov}(T', g(T'; \lambda)) &= (d_2^*/2)E(T' - b)^3/b + \mu_0^{-4}(d_3^*\sigma^4/2) \\ &\quad - \mu_0^{-3}(d_2^*\sigma^2 r(d_1^*)/2) - \mu_0^{-5}(d_2^*\sigma^2/2)^2 + o(1) \\ &= \mu_0^{-3}[d_2^*\sigma^2 r(d_1^*) - d_2^*E(X - \mu)^3/2] \\ &\quad + \mu_0^{-4}[(d_3^* + 3d_2^*)\sigma^4/2] + \mu_0^{-5}(2(d_2^*)^2\sigma^4) + o(1). \quad \square \end{aligned}$$

**PROOF OF THEOREM 4.** By (7.30) and Wald's lemma of Chow, Robbins and Teicher (1965)

$$\begin{aligned} \text{Var}(S_{T'} - \mu T') &= (\mu - d)^2 \text{Var}(T') - 2(\mu - d) \text{Cov}(T', g(T'; \lambda) + R') \\ &\quad + \text{Var}(R' + g(T'; \lambda)), \\ (\mu - d)^2 \text{Var}(T') &= \sigma^2 E T' + 2(\mu - d) \text{Cov}(T', R') \\ &\quad + 2(\mu - d) \text{Cov}(T', g(T', \lambda)) - \text{Var}(R' + g(T'; \lambda)). \end{aligned}$$

It follows from (7.32) and Lemmas 7–9 that

$$\begin{aligned} (\mu - d)^2 \text{Var}(T') &= \sigma^2 b + \sigma^2[r(d_1^*) + d_2^*(\sigma/\mu_0)^2/2]/\mu_0 \\ &\quad + \left[ \int_0^\infty x^2 dG(x, d_1^*) + (r(d_1^*))^2 - \mu_0^{-4}(d_2^*)^2\sigma^4/2 \right] \\ &\quad + 2C_3^* - \mu_0^{-1} d_2^*\sigma^2 \left[ (1 - \mu_0^{-2}\sigma^2) + 2 \sum_{n=1}^\infty P\{S_n - nd_1^* \leq 0\} \right] \\ &\quad + \mu_0^{-2} d_2^* [2\sigma^2 r(d_1^*) - E(X - \mu)^3] + \mu_0^{-3}(d_3^* + 3d_2^*)\sigma^4 \\ &\quad + \mu_0^{-4}(4(d_2^*)^2\sigma^4) + o(1), \end{aligned}$$

where

$$C_3^* = \int_0^\infty [ER(x, d_1^*) - r(d_1^*)] P\left\{ \max_{n \geq 0} (S_n - nd_1^*)^- \leq x \right\} dx.$$

Since

$$r(d_1^*) = (\mu_0^2 + \sigma^2)/(2\mu_0) - \sum_{n=1}^{\infty} n^{-1}E(S_n - nd_1^*)^-$$

and

$$\int_0^{\infty} [ER(x, d_1^*) - r(d_1^*)] dx = (r(d_1^*))^2 + \int_0^{\infty} x^2 dG(x, d_1^*)/2,$$

by algebra

$$\begin{aligned} & (\mu - d)^2 \text{Var}(T') - \sigma^2 b \\ &= \left[ \sigma^2 r(d_1^*)/\mu_0 + \int_0^{\infty} x^2 dG(x, d_1^*) + (r(d_1^*))^2 + 2C_3^* \right] \\ & \quad + \mu_0^{-3} d_2^* \sigma^4/2 - \mu_0^{-4} (d_2^*)^2 \sigma^4/2 - [\mu_0^{-1} d_2^* \sigma^2 - \mu_0^{-3} d_2^* \sigma^4] \\ & \quad - \mu_0^{-1} d_2^* \sigma^2 2 \sum_{n=1}^{\infty} P\{S_n - nd_1^* \leq 0\} \\ & \quad + [\mu_0^{-1} d_2^* \sigma^2 + \mu_0^{-3} d_2^* \sigma^4] - \mu_0^{-2} d_2^* \sigma^2 2 \sum_{n=1}^{\infty} n^{-1} E(S_n - nd_1^*)^- \\ & \quad - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3} (d_3^* + 3d_2^*) \sigma^4 + \mu_0^{-4} (4(d_2^*)^2 \sigma^4) + o(1) \\ &= \left[ \sigma^2 r(d_1^*)/\mu_0 + 2 \int_0^{\infty} x^2 dG(x, d_1^*) + 3(r(d_1^*))^2 \right] \\ & \quad - 2 \int_0^{\infty} [ER(x, d_1^*) - r(d_1^*)] P\left\{ \max_{n \geq 0} (S_n - nd_1^*)^- > x \right\} dx \\ & \quad - 2\mu_0^{-2} d_2^* \sigma^2 \left[ \mu_0 \sum_{n=1}^{\infty} P\{S_n - nd_1^* \leq 0\} + \sum_{n=1}^{\infty} n^{-1} E(S_n - nd_1^*)^- \right] \\ & \quad - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3} [11d_2^* \sigma^4/2 + d_3^* \sigma^4] \\ & \quad + 7\mu_0^{-4} (d_2^*)^2 \sigma^4/2 + o(1) \\ &= C_1^* - 2\mu_0^{-2} d_2^* \sigma^2 C_2^* - \mu_0^{-2} d_2^* E(X - \mu)^3 + \mu_0^{-3} d_3^* \sigma^4 \\ & \quad + \mu_0^{-4} d_2^* \sigma^4 (7d_2^* + 11\mu_0)/2 + o(1). \end{aligned}$$

This finishes the proof.  $\square$

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