Supplement to "Covariance Matrix Estimation for Stationary Time Series"

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In this document we give the proofs of Remark 5 and Lemma 9 of the main article, as well as a few remarks on Lemma 9. All the equation, theorem, lemma and remark numbers refer to the main article. The equations and remarks introduced in this document are numbered with an "S"-prefix.

Proof of Remark 5. Set $b'_T = 2\sum_{k=B_T+1}^{T-1} \gamma_k$. Define

$$g_{T,B_T}(\theta) = \pi^{-1} \sum_{k=B_T+1}^{T-1} \gamma_k \cos(k\theta).$$

Since $T^{-1}\sum_{k=1}^{B_T} k\gamma_k = O(B_T/T)$, if we can show that

$$\begin{aligned}\lim_{T\to\infty} P\Bigg\{ 2\pi \cdot \min_\theta \left[\hat{f}_{T,B_T}(\theta) - \mathbb{E}\,\hat{f}_{T,B_T}(\theta) - g_{T,B_T}(\theta) \right] \\ \le -\sqrt{\frac{B_T \log B_T}{2T}} - b'_T/5 \Bigg\} = 1,\end{aligned} \tag{S.1}$$

then (31) will follow by using (30) and similar arguments as those which have led to the lower bound in Theorem 2.

Let $k_T \in \mathbb{N}$ be such that $A^{k_T-1} \le B_T < A^{k_T}$. Define $\mathcal{D}_T = \{\theta \in [0,\pi] : \cos(A^{k_T}\theta) \ge 1/2\}$, then $2\pi \cdot g_{T,B_T}(\theta) \ge b'_T/5$ for $\theta \in \mathcal{D}_T$. Define $\lambda_{T,j} = 2\pi j\lfloor \log(B_T)^3 \rfloor / A^{k_T}$, and set $j_T = \max\{j : \lambda_{T,j} \le \pi\}$. Using the arguments of Liu and Wu (2010), if p is sufficiently large, we have

$$P\left[\min_{0\le j\le j_T} \sqrt{\frac{T}{B_T}} \frac{f_{T,B_T}(\lambda_{T,j}) - \mathbb{E} f_{T,B_T}(\lambda_{T,j})}{\sqrt{2} f(\lambda_{T,j})} \le \frac{x}{w_T} - z_T \right] \to 1 - e^{-e^{x/2}},$$

where $w_T = \sqrt{2\log j_T}$ and

$$z_T = (2\log j_T)^{1/2} - (8\log j_T)^{-1/2}(\log\log j_T + \log(4\pi)).$$

Since the spectral density is bounded away from zero, *i.e.*

$$\underline{f} := \min_\theta f(\theta) \geq \frac{1}{2\pi}\left(3 - 2\sum_{k=0}^{\infty} A^{-\alpha k}\right) \geq \frac{1}{4\pi},$$

and $j_T \geq B_T/[2\log(B_T)^3]$, it follows that

$$P\left\{\min_{0\leq j\leq j_T}[f_{T,B_T}(\lambda_{T,j}) - \mathbb{E}f_{T,B_T}(\lambda_{T,j})] \leq -\underline{f}\cdot\sqrt{\frac{2B_T\log B_T}{T}}\right\} \to 1.$$

Then (S.1) follows by noting that $\lambda_{T,j} \in \mathcal{D}_T$ for $0 \leq j \leq j_T$.

□

Proof of Lemma 9. Let $w_T = \lfloor m_T/2 \rfloor$, and split Q_T into two parts as

$$Q_{T,1} = \sum_{t=1}^{T} X_t \sum_{s=t-B_T}^{t-w_T-1} a_{s,t}X_s \quad \text{and} \quad Q_{T,2} = \sum_{t=1}^{T} X_t \sum_{s=t-w_T}^{t} a_{s,t}X_s,$$

where we make the convention that if a term X_s in the previous sum has the subscript $s \notin [1,T]$, then that term should be replaced by zero. Define $\tilde{Q}_{T,1}$ and $\tilde{Q}_{T,2}$ similarly. We consider $Q_{T,2}$ first. Write

$$\begin{aligned} Q_{T,2} - \tilde{Q}_{T,2} = &\sum_{t=1}^{T}(X_t - \tilde{X}_t)\sum_{s=t-w_T}^{t} a_{s,t}\mathcal{H}_{s-w_T}X_s + \sum_{s=1}^{T}(X_s - \tilde{X}_s)\sum_{t=s+1}^{s+w_T} a_{s,t}\tilde{X}_t \\ &+ \sum_{t=1}^{T}(X_t - \tilde{X}_t)\sum_{s=t-w_T}^{t} a_{s,t}(X_s - \mathcal{H}_{s-w_T}X_s) =: I_T + II_T + III_T. \end{aligned}$$

For the first term, write

$$I_T = \sum_{k=m_T+1}^{\infty}\sum_{t=1}^{T}\mathcal{P}_{t-k}X_t\sum_{s=t-w_T}^{t} a_{s,t}\mathcal{H}_{s-w_T}X_s.$$

Since $2w_T \leq m_T$, we know for each fixed $k > m_T + 1$,

$$\left(\mathcal{P}_{t-k}X_t\sum_{s=t-w_T}^{t} a_{s,t}\mathcal{H}_{s-w_T}X_s\right)_{1\leq t\leq T}$$

is a backward martingale difference sequence with respect to the filtration $(\mathcal{F}_{t-k})_{1\leq t\leq T}$. It follows that by (38)

$$\|I_T\|_{p/2} \leq \sum_{k=m_T+1}^{\infty}\mathcal{C}_{p/2}\sqrt{T}\delta_p(k)\,\mathcal{C}_p\sqrt{w_T\wedge B_T}\,\Theta_p \leq \mathcal{C}_p\mathcal{C}_{p/2}\Theta_p\sqrt{TB_T}\,\Theta_p(m_T).$$

Similarly we have $\|II_T\|_{p/2} \le C\sqrt{TB_T}\,\Theta_p(m_T)$. For the third term, using the arguments of Proposition 1 of Liu and Wu (2010), we have

$$\|III_T - \mathbb{E}III_T\|_{p/2} \le 2\sqrt{2}\,\mathcal{C}_{p/2}\mathcal{C}_p\sqrt{TB_T}\,[\Theta_p(w_T)\Delta_p(m_T) + \Theta_p(m_T)\Delta_p(w_T)]\,.$$

Now we consider $Q_{T,1}$. Observe that $Q_{T,1}$ is nonzero only when $B_T > w_T$. Write

$$\begin{aligned} Q_{T,1} - \tilde{Q}_{T,1} = & \sum_{t=1}^{T}(X_t - \tilde{X}_t)\sum_{s=t-B_T}^{t-w_T-1} a_{s,t}\tilde{X}_s + \sum_{s=1}^{T}(X_s - \tilde{X}_s)\sum_{t=s+w_T+1}^{s+B_T} a_{s,t}\tilde{X}_t \\ & + \sum_{t=1}^{T}(X_t - \tilde{X}_t)\sum_{s=t-B_T}^{t-w_T-1} a_{s,t}(X_s - \tilde{X}_s) =: IV_T + V_T + VI_T. \end{aligned}$$

Similarly as II_T and III_T, we have $\|V_T\|_{p/2} \le \mathcal{C}_{p/2}\mathcal{C}_p\Theta_p\sqrt{TB_T}\,\Theta_p(m_T)$ and

$$\|VI_T - \mathbb{E}(VI_T)\|_{p/2} \le 4\,\mathcal{C}_{p/2}\mathcal{C}_p\sqrt{TB_T}\,\Theta_p(m_T)\Delta_p(m_T).$$

Write the term IV_T as

$$IV_T = \sum_{k=m_T+1}^{\infty}\sum_{l=0}^{m_T}\sum_{t=1}^{T}\mathcal{P}_{t-k}X_t\sum_{s=t-B_T}^{t-w_T-1} a_{s,t}\mathcal{P}_{s-l}X_s.$$

For each fixed pair (k,l), if we remove the the pair (s,t) such that $t-k = s-l$ from the sum

$$\sum_{t=1}^{T}\mathcal{P}_{t-k}X_t\sum_{s=t-B_T}^{t-w_T-1} a_{s,t}\mathcal{P}_{s-l}X_s,$$

then by (38)

$$\left\|\sum_{t=1}^{T}\mathcal{P}_{t-k}X_t\sum_{t-B_T\le s<t-w_T,\, s-l\ne t-k} a_{s,t}\mathcal{P}_{s-l}X_s\right\|_{p/2} \le 2\,\mathcal{C}_{p/2}\mathcal{C}_p\,\sqrt{TB_T}\,\delta_p(k)\delta_p(l).$$

Therefore, it remains to deal with the term $\sum_{s=1}^{T}\mathcal{P}_{s-l}X_s\sum_{t\in\Lambda_s} a_{s,t}\mathcal{P}_{s-l}X_t$ for $0 \le l \le m_T$, where $\Lambda_s = [s+1+(w_T \vee (m_T - l))\,,(s+B_T)\wedge T]$. Since the sequence $(\mathcal{P}_{s-l}X_s\sum_{t\in\Lambda_s} a_{s,t}\mathcal{P}_{s-l}X_t)$ indexed by s is $(4B_T)$-dependent, and

$$\left\|\mathbb{E}_0\left(\sum_{s=1}^{4B_T}\mathcal{P}_{s-l}X_s\sum_{t\in\Lambda_s} a_{s,t}\mathcal{P}_{s-l}X_t\right)\right\|_{p/2} \le 2\cdot 4B_T\cdot\delta_p(l)\cdot\Theta_p(m_T);$$

we have by (38)

$$\left\|\mathbb{E}_0\left(\sum_{s=1}^{T}\mathcal{P}_{s-l}X_s\sum_{t=s+w_T+1}^{s+B_T} a_{s,t}\mathcal{P}_{s-l}X_t\right)\right\|_{p/2} \le 4\sqrt{2}\,\mathcal{C}_{p/2}\sqrt{TB_T}\delta_p(l)\Theta_p(m_T).$$

Putting these pieces together, the proof is complete. □

Remark S.1. If $\Theta_p(m) \asymp m^{-\alpha}$ for some $\alpha > 0$, then the bound becomes $C_p\sqrt{TB_T}m_T^{-\alpha}$. If $m_T = O(B_T)$, then this order of magnitude is optimal here, and cannot be improved in general. For example, consider the linear process $X_t = \sum_{s=0}^{\infty} a_s\epsilon_{t-s}$ and the quadratic form $Q_T = \sum_{1\le s<t\le T} X_tX_s\mathbf{1}_{0<t-s\le B_T}$, where $a_s = s^{-(1+\alpha)}$, and ϵ_s's are iid standard normal random variables. Observe that $Q_T - \tilde{Q}_T$ is also a quadratic form of ϵ_s's which can be written as

$$Q_T - \tilde{Q}_T = \sum_{-\infty<k\le l\le T} b_{k,l}\epsilon_k\epsilon_l,$$

which implies that

$$\left\|\mathbb{E}_0Q_T - \mathbb{E}_0\tilde{Q}_T\right\|^2 = 2\sum_{-\infty}^{T} b_{k,k}^2 + \sum_{-\infty<k<l\le T} b_{k,l}^2$$

Elementary but tedious calculations show that for $B_T < t < T - B_T$ and $\lfloor B_T/3\rfloor \le k \le \lfloor 2B_T/3\rfloor$, we have

$$b_{t,t-(m_T+1)-k} \ge \sum_{k=0}^{\lfloor B_T/3\rfloor} a_k \cdot \sum_{k=1}^{\lfloor B_T/3\rfloor} a_{m_T+k} \ge Cm_T^{-\alpha}.$$

It follows that $\|\mathbb{E}_0Q_T - \mathbb{E}_0\tilde{Q}_T\| \ge C\sqrt{TB_T}m_T^{-\alpha}$, namely the order of magnitude is achieved.

Remark S.2. A similar bound was obtained by Liu and Wu (2010):

$$\|\mathbb{E}_0Q_T - \mathbb{E}_0\tilde{Q}_T\|_{p/2} \le C_p\sqrt{TB_T}\,\Delta_p(m_T).$$

In term of the order of magnitude, our result is better, because $\Theta_p(m) \le \Delta_p(m)$. To be more precise, consider the condition $\Theta_p(m) = O(m^{-\alpha})$ for some $\alpha > 0$, which is the assumption we use for Theorem 4. Since $\Psi_p(m) \le \Theta_p(m)$, we have $\Psi_p(m) = O(m^{-\alpha})$. Conversely, if $\Psi_p(m) = O(m^{-\alpha+1/2})$, then $\Theta_p(m) = O(m^{-\alpha})$. A proof was given by Wu and Zhao (2008). Therefore, when using both $\Theta_p(m) = O(m^{-\alpha})$ and $\Psi_p(m) = O(m^{-\beta})$ as assumptions, we necessarily assume $\alpha > 0$ and $\alpha \le \beta \le \alpha + 1/2$ to avoid redundancy. Under these two conditions we have

$$\Delta_p(m) \le \sum_{k=0}^{\lfloor m^{\beta/(1+\alpha)}\rfloor} \min\{\delta_p(k), Cm^{-\beta}\} + \Theta_m(\lfloor m^{\beta/(1+\alpha)}\rfloor + 1) \le Cm^{-\alpha\beta/(1+\alpha)},$$

which implies that $\Delta_p(m) = O(m^{-\alpha\beta/(1+\alpha)})$. We shall give an example to show that the order cannot be improved. Define

$$\delta_p(k) = \begin{cases} k^{-(1+\alpha)} + 2^{-\beta n} & \text{if } k = 2^n \text{ for some } n \in \mathbb{N} \\ k^{-(1+\alpha)} & \text{otherwise.} \end{cases}$$

It is easily seen that $\Theta_p(m) \asymp m^{-\alpha}$ and $\Psi_p(m) \asymp m^{-\beta}$. Therefore,

$$\begin{aligned}
\Delta_p(2^n) &\geq \sum_{k=0}^{\infty} \min\{k^{-(1+\alpha)}, 2^{-\beta n}\} \\
&= \sum_{k=0}^{\lfloor 2^{\beta n/(1+\alpha)} \rfloor} \min\{k^{-(1+\alpha)}, 2^{-\beta n}\} + \Theta_p(\lfloor 2^{\beta n/(1+\alpha)} \rfloor + 1) \\
&\geq C\,(2^n)^{-\alpha\beta/(1+\alpha)}.
\end{aligned}$$

In particular, if we only put the condition on $\Theta_p(m)$, then the largest exponent γ such that $\Delta_p(m) = O(m^{-\gamma})$ for any sequence satisfying $\Theta_p(m) = O(m^{-\alpha})$ is $\gamma = \alpha^2/(1+\alpha)$.

References

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