Statistica Sinica: Supplement

PORTMANTEAU TEST AND SIMULTANEOUS INFERENCE FOR SERIAL COVARIANCES

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Supplementary Material

The supplementary file is organized as follows. We first collect in Section S1 some moment inequalities concerning the sums and quadratic forms of stationary processes, which might be useful for other studies. We then give the complements of Section 4.1 in Section S2, and the complements of Section 4.2 in Section S3, including the proofs of intermediate lemmas, as well as other theorems and corollaries from Section 2.1 and Section 2.2 respectively. In Section S4 we prove a normal comparison principle that is used in the proof of Theorem 1. We provide a sufficient condition for the summability of joint cumulants in Section S5. Some auxiliary results are collected in Section S6.

For the readiability and completeness of this document, the statements of Theorem 14 is repeated here. All the section, theorem, lemma and equation numbers refer to the main article. The sections, theorems, propositions, lemmas and equations introduced in this document are numbered with a "S"-prefix.

We list some notations here. The operator $\mathbb{E}_0$ is defined as $\mathbb{E}_0 X := X - \mathbb{E}X$ for any random variable X. For a vector $\boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$, let $|\boldsymbol{x}|$ be the Euclidean norm, $|\boldsymbol{x}|_\infty := \max_{1\le i\le d} |x_i|$, and $|\boldsymbol{x}|_\bullet := \min_{1\le i\le d} |x_i|$. For a square matrix A, $\rho(A)$ denotes the operator norm defined by $\rho(A) := \max_{|\boldsymbol{x}|=1} |A\boldsymbol{x}|$. Let us make some convention on the constants. We use C, c and $\mathcal{C}$ for constants. The notation $\mathcal{C}_p$ is reserved for the constant appearing in Burkholder's inequality, see (S.2). The values of C may vary from place to place, while the value of c is fixed within the statement and the proof of a theorem (or lemma). A constant with a symbolic subscript is used to emphasize the dependence of the value on the subscript.

S1 Some Useful Inequalities

We collect in Proposition S.1 some useful facts about physical dependence measures and martingale and m-dependence approximations. We expect that it will be useful in other asymptotic problems that involve sample covariances. Hence for convenience of other researchers, we provide explicit upper bounds.

We first introduce a moment inequality (S.1) which follows from the Burkholder inequality (see Burkholder, 1988). Let (D_i) be a martingale difference sequence and for every i, $D_i \in \mathcal{L}^p$, $p > 1$, then

$$\|D_1 + D_2 + \cdots + D_n\|_p^{p'} \le \mathcal{C}_p^{p'} \left(\|D_1\|_p^{p'} + \|D_2\|_p^{p'} + \cdots + \|D_n\|_p^{p'} \right), \tag{S.1}$$

where $p' = \min\{p, 2\}$, and the constant

$$\mathcal{C}_p = (p-1)^{-1} \text{ if } 1 < p < 2 \text{ and } = \sqrt{p-1} \text{ if } p \ge 2. \tag{S.2}$$

We note that when $p > 2$, the constant $\mathcal{C}_p$ in (S.1) equaled to $p-1$ in Burkholder (1988), and it was improved to $\sqrt{p-1}$ by Rio (2009).

Proposition S.1. *1. Assume $\mathbb{E}X_i = 0$ and $p > 1$. Recall that $p' = \min(p, 2)$.*

$$\|\mathcal{P}^0 X_i\|_p \le \delta_p(i) \quad \textit{and} \quad \|\mathcal{P}_0 X_i\|_p \le \delta_p(i) \tag{S.3}$$

$$\kappa_p := \|X_0\|_p \le \mathcal{C}_p \Psi_p \tag{S.4}$$

$$\left\| \sum_{i=1}^n c_i X_i \right\|_p \le \mathcal{C}_p A_n \Theta_p, \;\; \textit{where } A_n = \left(\sum_{i=1}^n |c_i|^{p'} \right)^{1/p'} \tag{S.5}$$

$$|\gamma_k| \le \zeta_2(k), \quad \textit{where } \zeta_p(k) := \sum_{j=0}^\infty \delta_p(j)\delta_p(j+k) \tag{S.6}$$

$$\left\| \sum_{i=1}^n (X_{i-k}X_i - \gamma_k) \right\|_{p/2} \le 2\mathcal{C}_{p/2}\kappa_p \Theta_p \sqrt{n}, \quad \textit{when } p \ge 4 \tag{S.7}$$

$$\left\| \sum_{i,j=1}^n c_{i,j}(X_i X_j - \gamma_{i-j}) \right\|_{p/2} \le 4\mathcal{C}_{p/2}\mathcal{C}_p \Theta_p^2 B_n \sqrt{n}, \quad \textit{when } p \ge 4 \tag{S.8}$$

where $B_n^2 = \max\{\max_{1\le i\le n} \sum_{j=1}^n c_{i,j}^2, \max_{1\le j\le n}\sum_{i=1}^n c_{i,j}^2\}$.

2. For $m \ge 0$, define $\tilde{X}_i = \mathcal{H}_{i-m} X_i$. For $p > 1$, let $\tilde{\delta}_p(\cdot)$ be the physical dependence measures for the sequence $(\tilde{X}_i)$. Then

$$\tilde{\delta}_p(i) \le \delta_p(i) \tag{S.9}$$

$$\|X_0 - \tilde{X}_0\|_p \le \mathcal{C}_p \Psi_p(m+1) \tag{S.10}$$

$$\left\| \sum_{i=1}^n c_i (X_i - \tilde{X}_i) \right\|_p \le \mathcal{C}_p A_n \Theta_p(m+1) \tag{S.11}$$

$$\left\| \sum_{i=k+1}^n \left(X_{i-k}X_i - \gamma_k - \tilde{X}_{i-k}\tilde{X}_i + \tilde{\gamma}_k \right) \right\|_p \le 4\mathcal{C}_p (n-k)^{1/p'} \kappa_{2p} \Delta_{2p}(m+1). \tag{S.12}$$

Proof of Proposition S.1. The inequalities (S.3) and (S.9) are obtained by the first principle. Since $X_{i-k} = \sum_{j\in\mathbb{Z}} \mathcal{P}^j X_{i-k}$ and $X_i = \sum_{j\in\mathbb{Z}} \mathcal{P}^j X_i$, we have

$$|\gamma_k| = \left| \sum_{j=-k}^{\infty} \mathbb{E}\left[(\mathcal{P}^{-j}X_0)(\mathcal{P}^{-j}X_k)\right] \right| \leq \delta_2(j)\delta_2(j+k) \leq \zeta_k,$$

which proves (S.6). For (S.8), it can be similarly proved as Proposition 1 of Liu and Wu (2010), and (S.11) was given by Lemma 1 of the same paper. (S.5) is a special case of (S.11). Define $Y_i = X_{i-k}X_i$, then (Y_i) is also a stationary process of the form (9). By Hölder's inequality, $\|Y_i - \Omega_0(Y_i)\|_{p/2} \leq 2\kappa_p[\delta_p(i) + \delta_p(i-k)]$. Applying (S.5) to (Y_i), we obtain (S.7). To see (S.10), we first write $X_m - \tilde{X}_m = \sum_{j=1}^{\infty} \mathcal{P}_{-j} X_m$. Since $\|\mathcal{P}_{-j}X_m\|_p \leq \delta_p(m+j)$, and $(\mathcal{P}_{-j}X_m)_{j\geq 1}$ is a martingale difference sequence, by (S.1), we have

$$\|X_0 - \tilde{X}_0\|_p^{p'} \leq \mathcal{C}_p^{p'} \sum_{j=1}^{\infty} \|\mathcal{P}_{-j}X_m\|_p^{p'} \leq \mathcal{C}_p^{p'} \sum_{j=1}^{\infty} [\delta_p(m+j)]^{p'} = \mathcal{C}_p^{p'} [\Psi_p(m+1)]^{p'}.$$

The above argument also leads to (S.4). Using a similar argument as in the proof of Theorem 2 of Wu (2009), we can show (S.12). Details are omitted. □

S2 Complements of Section 4.1

We prove the five intermediate steps in Section S2.1∼S2.5, and Theorem 2 in Section S2.6.

S2.1 Step 1: m-dependence approximation

Proof of Lemma 8. Recall that $m_n = \lfloor n^\beta \rfloor$ with $\eta < \beta < 1$. We claim

$$\left\| R_{n,k} - \tilde{R}_{n,k} \right\|_{p/2} \leq 6\,\mathcal{C}_{p/2}\Theta_p\Theta_p(m_n - k + 1)\cdot\sqrt{n}. \tag{S.13}$$

It follows that for any $\lambda > 0$

$$P\left(\max_{1\le k\le s_n}\left|R_{n,k}-\tilde{R}_{n,k}\right| > \lambda\sqrt{n/\log s_n}\right) \le \frac{(\log s_n)^{p/4}}{n^{p/4}\lambda^{p/2}}\sum_{k=1}^{s_n}\|R_{n,k}-\tilde{R}_{n,k}\|_{p/2}^{p/2}$$

$$\le C_p\lambda^{-p/2}s_n(\log s_n)^{p/4}n^{-\alpha\beta p/2} \le C_p\lambda^{-p/2}n^{\eta-\alpha\beta p/2}(\log n)^{p/4}.$$

Therefore, if $\alpha p/2 > \eta$, then there exists a β such that $\eta < \beta < 1$ and $\eta - \alpha\beta p/2 < 0$, and hence the preceding probability goes to zero as $n \to \infty$. The proof of Lemma 8 is complete.

We now prove claim (S.13). For each $1 \le k \le s_n$, we have

$$\|R_{n,k}-\tilde{R}_{n,k}\|_{p/2} \le \left\|\sum_{i=k+1}^{n}(X_{i-k}-\tilde{X}_{i-k})\tilde{X}_i\right\|_{p/2} + \left\|\sum_{i=k+1}^{n}(\mathcal{H}_{i-m_n}X_{i-k})(X_i-\tilde{X}_i)\right\|_{p/2}$$
$$+\left\|\sum_{i=k+1}^{n}\mathbb{E}_0\left[(X_{i-k}-\mathcal{H}_{i-m_n}X_{i-k})(X_i-\tilde{X}_i)\right]\right\|_{p/2}$$

Observe that $(\tilde{X}_i\mathcal{P}_{i-k-j}X_{i-k})_{1\le i\le n}$ is a backward martingale difference sequence with respect to $\mathcal{F}_{i-k-j}$ if $j > m_n$, so by the inequality (S.1),

$$\begin{aligned}\left\|\sum_{i=k+1}^{n}(X_{i-k}-\tilde{X}_{i-k})\tilde{X}_i\right\|_{p/2} &\le \sum_{j=m+1}^{\infty}\left\|\sum_{i=k+1}^{n}\tilde{X}_i\mathcal{P}_{i-k-j}X_{i-k}\right\|_{p/2}\\ &\le \sum_{j=m+1}^{\infty}\sqrt{n}\mathcal{C}_{p/2}\|\tilde{X}_{j+k}\mathcal{P}_0X_j\|_{p/2}\\ &\le \mathcal{C}_{p/2}\Theta_p\Theta_p(m_n+1)\cdot\sqrt{n}.\end{aligned}$$

Similarly we have $\|\sum_{i=k+1}^{n}(\mathcal{H}_{i-m_n}X_{i-k})(X_i-\tilde{X}_i)\|_{p/2} \le \sqrt{n}\mathcal{C}_{p/2}\Theta_p\Theta_p(m_n+1)$. Similarly as (S.11), we get $\|\tilde{X}_{i-k}-\mathcal{H}_{i-m_n}X_{i-k}\|_p \le \Theta_p(m_n-k+1)$. Let $Y_{n,i} := (X_{i-k} - \mathcal{H}_{i-m_n}X_{i-k})(X_i-\tilde{X}_i)$. Then

$$\|Y_{n,i}-\Omega_0(Y_{n,i})\|_{p/2} \le 2\left[\delta_p(i)\Theta_p(m_n-k+1)+\delta_p(i-k)\Theta_p(m_n+1)\right].$$

Therefore, by (S.5), it follows that

$$\left\| \sum_{i=k+1}^{n} \mathbb{E}_0 \left[(X_{i-k} - \mathcal{H}_{i-m_n} X_{i-k})(X_i - \tilde{X}_i) \right] \right\|_{p/2} \leq 4\,\mathcal{C}_{p/2} \Theta_p \Theta_p (m_n - k + 1) \cdot \sqrt{n},$$

and the proof of (S.13) is complete. □

S2.2 Step 2: Throw out small blocks

In this section, as well as many other places in this article, we often need to split an integer interval $[s,t] = \{s, s+1, \ldots, t\} \subset \mathbb{N}$ into consecutive blocks $\mathcal{B}_1, \ldots, \mathcal{B}_w$ with the size m. Since $s - t + 1$ may not be a multiple of m, we make the convention that unless the size of the last block is specified clearly, it has the size $m \leq |\mathcal{B}_w| < 2m$, and all the other ones have the same size m.

Proof of Lemma 9. It suffices to show that for any $\lambda > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{s_n} P\left(\left| \sum_{j=1}^{w_n} V_{k,j} \right| \geq \lambda \sqrt{\frac{n}{\log s_n}} \right) = 0.$$

Observe that $V_{k,j}, 1 \leq j \leq w_n$, are independent. By (S.7), $\|V_{k,j}\| \leq 2|K_j|^{1/2}\kappa_4\Theta_4$. By Corollary 1.6 of Nagaev (1979), for any $M > 1$, there exists a constant $C_M > 1$ such that

$$\begin{aligned}
&P\left(\left| \sum_{j=1}^{w_n} V_{k,j} \right| \geq \lambda \sqrt{\frac{n}{\log s_n}} \right) \\
&\leq \sum_{j=1}^{w_n} P\left(|V_{k,j}| \geq C_M^{-1} \lambda \sqrt{n/\log s_n} \right) + \left(\frac{4e^2\kappa_4^2\Theta_4^2 \sum_{j=1}^{w_n} |K_j|}{C_M^{-1}\lambda^2 n/\log s_n} \right)^{C_M/2} \\
&\leq \sum_{j=1}^{w_n} P\left(|V_{k,j}| \geq C_M^{-1} \lambda \sqrt{n/\log n} \right) + C_M \left(n^{\beta-\gamma} \log n \right)^{C_M/2} \\
&\leq \sum_{j=1}^{w_n} P\left(|V_{k,j}| \geq C_M^{-1} \sqrt{n/\log n} \right) + n^{-M}.
\end{aligned} \tag{S.14}$$

where we resolve the constant λ into the constant C_M in the last inequality. It remains to show that

$$\lim_{n\to\infty}\sum_{k=1}^{s_n}\sum_{j=1}^{w_n} P\left(|V_{k,j}| \geq q_1\delta\phi_n\right) = 0, \text{ where } \phi_n = \sqrt{\frac{n}{\log n}}, \tag{S.15}$$

holds for any $\delta > 0$, where q_1 is the smallest integer such that $\beta^{q_1} < \min\{(p-4)/p, (p-2-2\eta)/(p-2)\}$. This choice of q_1 will be explained later. We adopt the technique of successive m-dependence approximations from Liu and Wu (2010) to prove (S.15).

For $q \geq 1$, set $m_{n,q} = \lfloor n^{\beta^q}\rfloor$. Define $X_{i,q} = \mathcal{H}_{i-m_{n,q}}X_i$, $\gamma_{k,q} = \mathbb{E}(X_{0,q}X_{k,q})$, and

$$V_{k,j,q} = \sum_{i\in K_j, i>k}(X_{i-k,q}X_{i,q} - \gamma_{k,q}).$$

In particular, $m_{n,1}$ is same as m_n defined in Step 2, and $V_{k,j,1} = V_{k,j}$. Without loss of generality assume $s_n \leq \lfloor n^{\eta}\rfloor$. Let q_0 be such that $\beta^{q_0+1} \leq \eta < \beta^{q_0}$. We first consider the difference between $V_{k,j,q}$ and $V_{k,j,q+1}$ for $1 \leq q < q_0$. Split the block K_j into consecutive small blocks $\mathcal{B}_1, \ldots, \mathcal{B}_{w_{n,q}}$ with size $2m_{n,q}$. Define

$$V^{(0)}_{k,j,q,t} = \sum_{i\in\mathcal{B}_t}(X_{i-k,q}X_{i,q} - \gamma_{k,q}) \quad \text{and} \quad V^{(1)}_{k,j,q,t} = \sum_{i\in\mathcal{B}_t}(X_{i-k,q+1}X_{i,q+1} - \gamma_{k,q+1}). \tag{S.16}$$

Observe that $V^{(0)}_{k,j,q,t_1}$ and $V^{(0)}_{k,j,q,t_2}$ are independent if $|t_1 - t_2| > 1$. Similar as (S.14), for any $M > 1$, there exists a constant $C_M > 1$ such that, for sufficiently large n,

$$\begin{aligned} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right) &= P\left[\left|\sum_{t=1}^{w_{n,q}}\left(V^{(0)}_{k,j,q,t} - V^{(1)}_{k,j,q,t}\right)\right| \geq \delta\phi_n\right] \\ &\leq \sum_{t=1}^{w_{n,q}} P\left(\left|V^{(0)}_{k,j,q,t} - V^{(1)}_{k,j,q,t}\right| \geq C_M^{-1}\phi_n\right) + n^{-M}. \end{aligned} \tag{S.17}$$

Similarly as (S.13), we have $\left\|V^{(0)}_{k,j,q,t} - V^{(1)}_{k,j,q,t}\right\|_{p/2} \leq C_p|\mathcal{B}_t|^{1/2}m_{n,q+1}^{-\alpha}$. It follows that

$$\sum_{k=1}^{s_n}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right) \leq C_{p,M}n^{\eta}n^{1-\gamma}\left(n^{-M} + \frac{n^{\gamma}m_{n,q}^{p/4}m_{n,q+1}^{-\alpha p/2}}{m_{n,q}(n/\log n)^{p/4}}\right)$$

$$\leq C_{p,M}\left(n^{\eta+1-\gamma-M} + n^{\eta} n^{1-p/4} m_{n,q}^{p/4-1-\alpha\beta p/2}\right).$$

Under the condition (16), there exists a $0 < \beta < 1$, such that

$$\sum_{k=1}^{s_n}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right)$$
$$\leq C_{p,M}\left(n^{\eta+1-\gamma-M} + n^{\eta+1-p/4+\beta^q(p/4-1-\alpha\beta p/2)}\right) \to 0.$$

Recall that q_1 is the smallest integer such that $\beta^{q_1} < \min\{(p-4)/p, (p-2-2\eta)/(p-2)\}$. We now consider the difference between $V_{k,j,q}$ and $V_{k,j,q+1}$ for $q_0 \leq q < q_1$. The problem is more complicated than the preceding case $1 \leq q < q_0$, since now it is possible that $m_{n,q} < k$ for some $1 \leq k \leq s_n$. We consider three cases.

Case 1: $k \geq 2m_{n,q}$. Partition the block K_j into consecutive smaller blocks $\mathcal{B}_1, \ldots, \mathcal{B}_{w_{n,q}}$ with same size $m_{n,q}$. Define $V^{(0)}_{k,j,q,t}$ and $V^{(1)}_{k,j,q,t}$ as in (S.16). Observe that the sequence $\left(V^{(0)}_{k,j,q,t} - V^{(1)}_{k,j,q,t}\right)_{t \text{ is odd}}$ is a martingale difference sequence with respective to the filtration $(\xi_t := \langle \epsilon_l : l \leq \max\{\mathcal{B}_t\}\rangle)_{t \text{ is odd}}$, and so is the sequence and filtration labelled by even t. Set $\xi_0 = \langle \epsilon_l : l < \min\{\mathcal{B}_1\}\rangle$ and $\xi_{-1} = \langle \epsilon_l : l < \min\{\mathcal{B}_1\} - m_{n,q}\rangle$. For each $1 \leq t \leq w_{n,q}$, define

$$\mathcal{V}^{(l)}_t = \mathbb{E}\left[\left(V^{(l)}_{k,j,q,t}\right)^2 |\xi_{t-2}\right] = \sum_{i_1,i_2 \in \mathcal{B}_t} X_{i_1-k,q+l} X_{i_2-k,q+l} \gamma_{i_1-i_2,q+l}$$

for $l = 0, 1$. By Lemma 1 of Haeusler (1984), for any $M > 1$, there exists a constant $C_M > 1$ such that

$$\begin{aligned} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right) &\leq \sum_{t=1}^{w_{n,q}} P\left(\left|V^{(0)}_{k,j,q,t} - V^{(1)}_{k,j,q,t}\right| \geq \sqrt{\frac{n}{(\log n)^3}}\right) + n^{-M} \\ &+ \sum_{l=0,1} 2\left\{P\left[\sum_{t \text{ is odd}} \mathcal{V}^{(l)}_t \geq \frac{C_M^{-1} n}{(\log n)^2}\right] + P\left[\sum_{t \text{ is even}} \mathcal{V}^{(l)}_t \geq \frac{C_M^{-1} n}{(\log n)^2}\right]\right\}. \end{aligned} \tag{S.18}$$

By (S.6), $\sum_{k\in\mathbb{Z}}|\gamma_{k,q+l}|^2 \le \Theta_2^2$, and hence by (S.8), $\|\mathcal{V}_t^{(l)}\|_{p/2} \le C_p m_{n,q}^{1/2}$. Observe that $\mathcal{V}_{t_1}^{(0)}$ and $\mathcal{V}_{t_1}^{(0)}$ are independent if $|t_1 - t_2| > 1$, so similarly as (S.14), we have

$$\begin{aligned}
P\left[\sum_{t \text{ is odd}} \mathcal{V}_t^{(l)} \ge \frac{C_M^{-1} n}{(\log n)^2}\right] &\le n^{-M} + \sum_{t \text{ is odd}} P\left[\mathcal{V}_t^{(l)} \ge \frac{C_M^{-2} n}{(\log n)^2}\right] \\
&\le n^{-M} + C_{p,M}\cdot w_{n,q}\cdot n^{-p/2}(\log n)^p \cdot m_{n,q}^{p/4}.
\end{aligned}$$

The same inequality holds for the sum over even t. For the first term in (S.18), we claim that

$$\left\|V_{k,j,q,t}^{(0)} - V_{k,j,q,t}^{(1)}\right\|_p \le C_p \cdot m_{n,q}^{1/2}\cdot m_{n,q+1}^{-\alpha}, \tag{S.19}$$

which together with the preceding two inequalities implies that

$$P\left(|V_{k,j,q} - V_{k,j,q+1}| \ge \delta\phi_n\right) \le C_{p,M}\, w_{n,q}\cdot n^{-p/2}(\log n)^{3p/2}\left(m_{n,q}^{p/2}\cdot m_{n,q+1}^{-\alpha p} + m_{n,q}^{p/4}\right) + n^{-M}.$$

It follows that under condition (16), there exists a $0<\beta<1$ such that

$$\begin{aligned}
&\sum_{k=2m_{n,q}}^{s_n}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \ge \delta\phi_n\right) \\
&\quad\le n^{1+\eta-M} + C_{p,M}\cdot n^{1+\eta-p/2}(\log n)^{3p/2}\left[n^{\beta^q(p/2-1-\alpha\beta p)} + n^{\beta^q(p/4-1)}\right] = o(1).
\end{aligned} \tag{S.20}$$

Case 2: $k \le m_{n,q+1}/2$. Partition the block K_j into consecutive smaller blocks $\mathcal{B}_1,\ldots,\mathcal{B}_{w_{n,q}}$ with size $3m_{n,q}$. Define $V_{k,j,q,t}^{(0)}$ and $V_{k,j,q,t}^{(1)}$ as in (S.16). Similarly as (S.13), we have

$$\left\|V_{k,j,q,t}^{(0)} - V_{k,j,q,t}^{(1)}\right\|_{p/2} \le C_p\cdot m_{n,q}^{1/2}\cdot m_{n,q+1}^{-\alpha}.$$

Similar as (S.17), for any $M>1$, there exist a constant $C_M>1$ such that

$$\begin{aligned}
P\left(|V_{k,j,q} - V_{k,j,q+1}| \ge \delta\phi_n\right) &\le \sum_{t=1}^{w_{n,q}} P\left(\left|V_{k,j,q,t}^{(0)} - V_{k,j,q,t}^{(1)}\right| \ge C_M^{-1}\phi_n\right) + n^{-M} \\
&\le n^{-M} + C_{p,M}\cdot w_{n,q}\cdot n^{-p/4}(\log n)^{p/4}\cdot m_{n,q}^{p/4}\cdot m_{n,q+1}^{-\alpha\beta p/2}.
\end{aligned}$$

It follows that that under condition (16), there exists a $0 < \beta < 1$ such that

$$\begin{aligned}&\sum_{k=1}^{m_{n,q+1}/2}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right)\\&\leq n^{1+\eta-M} + C_{p,M}\cdot n^{1-p/4}(\log n)^{p/4}\cdot\left(n^{\beta^q}\right)^{p/4-\alpha\beta p/2} = o(1).\end{aligned} \tag{S.21}$$

Case 3: $m_{n,q+1}/2 < k < 2m_{n,q}$. We use the same argument as in Case 2. But this time we claim that

$$\left\|V_{k,j,q,t}^{(0)} - V_{k,j,q,t}^{(1)}\right\|_{p/2} \leq C_p\left[m_{n,q}^{1/2}\cdot m_{n,q+1}^{-\alpha} + m_{n,q}\zeta_p(k)\right], \tag{S.22}$$

where $\zeta_p(k)$ is defined in (S.6). Since $\sum_{k=m}^{\infty}[\zeta_p(k)]^{p/2} \leq \left[\sum_{k=m}^{\infty}\zeta_p(k)\right]^{p/2} = O(m^{-\alpha p/2})$, under the condition (12), there exist constants $C_{p,M} > 1$ and $0 < \beta < 1$ such that for M large enough

$$\begin{aligned}&\sum_{k>m_{n,q+1}/2}^{2m_{n,q}-1}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right) \leq C_{p,M}\cdot n^{1-p/4}(\log n)^{p/4} m_{n,q}^{p/4-\alpha\beta p/2}\\&\quad + n^{1+\eta-M} + C_{p,M}\cdot n^{1-p/4}(\log n)^{p/4}\cdot m_{n,q}^{p/2-1}\sum_{k>m_{n,q+1}/2}^{2m_{n,q}-1}[\zeta_p(k)]^{p/2}\\&\leq n^{1+\eta-M} + C_{p,M}\cdot n^{1-p/4}(\log n)^{p/4}\cdot m_{n,q}^{p/2-1-\alpha\beta p/2} = o(1).\end{aligned} \tag{S.23}$$

Alternatively, if we use the bound from (S.12), $\left\|V_{k,j,q,t}^{(0)} - V_{k,j,q,t}^{(1)}\right\|_{p/2} \leq C_p m_{n,q}^{1/2}\cdot m_{n,q+1}^{-\alpha'}$, it is still true that under condition (12), there exist constants $C_{p,M} > 1$ and $0 < \beta < 1$ such that for M large enough

$$\begin{aligned}&\sum_{k>m_{n,q+1}/2}^{2m_{n,q}-1}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right)\\&\leq n^{1+\eta-M} + C_{p,M}\cdot n^{1-p/4}(\log n)^{p/4}\cdot m_{n,q}^{p/2-1-\alpha'\beta p/2} = o(1).\end{aligned} \tag{S.24}$$

Combine (S.20), (S.21), (S.23) and (S.24), we have shown that

$$\lim_{n\to\infty}\sum_{k=1}^{s_n}\sum_{j=1}^{w_n} P\left(|V_{k,j,q} - V_{k,j,q+1}| \geq \delta\phi_n\right) = 0. \tag{S.25}$$

for $1 \leq q < q_1$. Therefore, to prove (S.15), it suffices to show

$$\lim_{n\to\infty} \sum_{k=1}^{s_n} \sum_{j=1}^{w_n} P\left(|V_{k,j,q_1}| \geq \delta\phi_n\right) = 0 \tag{S.26}$$

By considering two cases (i) $2m_{n,q_1} \leq k \leq s_n$ and (ii) $1 \leq k < 2m_{n,q_1}$ under the condition $\beta^{q_1} < \min\{(p-4)/p, (p-2-2\eta)/(p-2)\}$, and using similar arguments as those in proving (S.25), we can obtain (S.26). The proof of Lemma 9 is complete.

We now turn to the proof of the two claims (S.19) and (S.22). For (S.22), we have

$$\begin{aligned}
\left\|V^{(0)}_{k,j,q,t} - V^{(1)}_{k,j,q,t}\right\|_{p/2} \leq & \left\|\sum_{i\in\mathcal{B}_t} (X_{i-k,q} - X_{i-k,q+1}) X_{i,q+1}\right\|_{p/2} \\
& + \left\|\sum_{i\in\mathcal{B}_t} \mathbb{E}_0\left[X_{i-k,q+1}(X_{i,q} - X_{i,q+1})\right]\right\|_{p/2} \\
& + \left\|\sum_{i\in\mathcal{B}_t} \mathbb{E}_0\left[(X_{i-k,q} - X_{i-k,q+1})(X_{i,q} - X_{i,q+1})\right]\right\|_{p/2} \\
=: & I + II + III.
\end{aligned}$$

Similarly as in the proof of (S.13), we have

$$I \leq \mathcal{C}_{p/2}\Theta_p\Theta_p(m_{n,q+1}+1)\cdot\sqrt{3m_{n,q}} \quad \text{and} \quad III \leq 4\,\mathcal{C}_{p/2}\Theta_p\Theta_p(m_{n,q+1}+1)\cdot\sqrt{3m_{n,q}}.$$

For the second term II, write

$$\mathbb{E}_0\left[X_{i-k,q+1}(X_{i,q} - X_{i,q+1})\right] = \sum_{l_1=0}^{m_{n,q+1}} \sum_{l_2=m_{n,q+1}+1}^{m_{n,q}} \mathbb{E}_0\left[(\mathcal{P}_{i-k-l_1}X_{i-k})(\mathcal{P}_{i-l_2}X_i)\right].$$

For a pair (l_1, l_2) such that $i-k-l_1 \neq i-l_2$, by the inequality (S.1), we have

$$\left\|\sum_{i\in\mathcal{B}_t} (\mathcal{P}_{i-k-l_1}X_{i-k})(\mathcal{P}_{i-l_2}X_i)\right\|_{p/2} \leq \mathcal{C}_{p/2}\delta_p(l_1)\delta_p(l_2)\cdot\sqrt{3m_{n,q}}.$$

For the pairs (l_1, l_2) such that $i-k-l_1 = i-l_2$, by the triangle inequality

$$\begin{aligned}&\left\|\sum_{i\in\mathcal{B}_t}\sum_{l=0}^{m_{n,q+1}}\mathbb{E}_0\left[(\mathcal{P}_{i-k-l}X_{i-k})(\mathcal{P}_{i-k-l}X_i)\right]\right\|_{p/2}\\ \leq&3m_{n,q}\cdot 2\sum_{l=0}^{m_{n,q+1}}\delta_p(l)\delta_p(k+l)\leq 6m_{n,q}\zeta_p(k).\end{aligned}$$

Putting these pieces together, the proof of (S.22) is complete. The key observation in proving (S.19) is that since $k \geq 2m_{n,q}$, $X_{i-k,q}$ and $X_{i,q}$ are independent, hence the product $X_{i-k,q}X_{i,q}$ has finite p-th moment. The rest of the proof is similar to that of (S.22). Details are omitted. □

Remark S.1. Condition (12) is only used to deal with Case 3, while (16) suffices for the rest of the proof. In fact, for linear processes, one can show that the term $m_{n,q}\zeta_p(k)$ in (S.22) can be removed, so we have (S.23) under condition (16) and do not need (S.24). So (16) suffices for Theorem 1. Furthermore, for nonlinear processes with $\delta_p(k) = O\left[k^{-(1/2+\alpha)}\right]$, the term $m_{n,q}\zeta_p(k)$ can also be removed from (S.22). Details are omitted.

S2.3 Step 3: Truncate sums over large blocks

Proof of Lemma 10. We need to show for any $\lambda > 0$

$$\lim_{n\to\infty}\sum_{k=1}^{s_n}P\left(\left|\sum_{j=1}^{w_n}(U_{k,j}-\bar{U}_{k,j})\right|\geq\lambda\sqrt{\frac{n}{\log s_n}}\right)=0.$$

Using (S.7), elementary calculation gives

$$\left\|U_{k,j}-\bar{U}_{k,j}\right\|^2\leq\frac{\mathbb{E}|U_{k,j}|^{p/2}}{(\sqrt{n}/\log s_n)^{p/2-2}}\leq\frac{(2\mathcal{C}_{p/2}\kappa_p\Theta_p)^{p/2}|H_j|^{p/4}(\log s_n)^{3(p-4)/2}}{n^{(p-4)/4}}. \tag{S.27}$$

Similarly as (S.14), for any $M > 1$, there exists a constant $C_M > 1$ such that

$$\begin{aligned} P\left(\left|\sum_{j=1}^{w_n}(U_{k,j} - \bar{U}_{k,j})\right| \ge \lambda\sqrt{\frac{n}{\log s_n}}\right) &\le \sum_{j=1}^{w_n} P\left(|U_{k,j} - \bar{U}_{k,j}| \ge C_M^{-1}\lambda\sqrt{\frac{n}{\log s_n}}\right) \\ &\quad + \left(\frac{C_p \sum_{j=1}^{w_n} |H_j|^{p/4}(\log n)^{3p/2}}{C_M^{-1}\lambda^2 n^{p/4}}\right)^{C_M/2} \\ &\le \sum_{j=1}^{w_n} P\left(|U_{k,j} - \bar{U}_{k,j}| \ge C_M^{-1}\sqrt{\frac{n}{\log s_n}}\right) + n^{-M}. \end{aligned}$$

Therefore, it suffices to show that for any $\delta > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{s_n}\sum_{j=1}^{w_n} P\left(|U_{k,j} - \bar{U}_{k,j}| \ge \delta\sqrt{\frac{n}{\log n}}\right) = 0.$$

Since we can use the same arguments as those for (S.15), Lemma 10 follows. $\square$

S2.4 Step 4: Compare covariance structures

Lemma 11 is obtained by a simple application of the Bernstein's in equality, so we omit the proof. The following lemma is an intermediate step for proving Lemma 12.

Lemma S.2. *Assume $X_i \in \mathcal{L}^4$, $\mathbb{E}X_0 = 0$, and $\Theta_4 < \infty$. Assume $l_n \to \infty$, $k_n \to \infty$, $\check{m}_n < \lfloor k_n/3 \rfloor$ and $h \ge 0$. Define $S_{n,k} = \sum_{i=1}^{l_n}(X_{i-k}X_i - \gamma_k)$. Then*

$$|\mathbb{E}\left(S_{n,k_n}S_{n,k_n+h}\right)/l_n - \sigma_h| \le \Theta_4^3\left(16\Delta_4(\check{m}_n+1) + 6\Theta_4\sqrt{\check{m}_n/l_n} + 4\Psi_4(\check{m}_n+1)\right).$$

Proof. Let $\check{X}_i = \mathcal{H}_{i-\check{m}_n}^i X_i$, then $\check{X}_i$ and $\check{X}_{i-k_n}$ are independent, because $\check{m}_n \le \lfloor k_n/3 \rfloor$. Define $\check{S}_{n,k} = \sum_{i=1}^{l_n} \check{X}_{i-k}\check{X}_i$. By (S.12), we have for any $k \ge 0$,

$$\left\|(S_{n,k} - \check{S}_{n,k})/\sqrt{l_n}\right\| \le 4\kappa_4\Delta_4(\check{m}_n+1).$$

By (S.7), $\|S_{n,k}/\sqrt{l_n}\| \le 2\kappa_4\Theta_4$ for any $k \ge 0$, and it follows that

$$\begin{aligned}
&\left|\mathbb{E}(S_{n,k_n}, S_{n,k_n+h}) - \mathbb{E}(\check{S}_{n,k_n}\check{S}_{n,k_n+h})\right| \\
&\le \|S_{n,k_n} - \check{S}_{n,k_n}\| \cdot \|S_{n,k_n+h}\| + \|\check{S}_{n,k_n}\| \cdot \|S_{n,k_n+h} - \check{S}_{n,k_n+h}\| \\
&\le 16 l_n \kappa_4^2 \Theta_4 \Delta_4(\check{m}_n + 1).
\end{aligned} \tag{S.28}$$

For any $k > 3\check{m}_n$, define $M_{n,k} = \sum_{j=1}^{l_n} D_j$, where

$$D_j = \sum_{i=j}^{j+\check{m}_n} \check{X}_{i-k}\mathcal{P}^j\check{X}_i = \sum_{q=0}^{\check{m}_n} X_{j+q-k}\mathcal{P}^j X_{j+q}.$$

Observe that $\mathcal{P}^j\check{X}_{j+q}$ and $\check{X}_{j+q-k}$ are independent, we have

$$\begin{aligned}
\|\check{S}_{n,k} - M_{n,k}\| &= \left\| \sum_{i=1}^{l_n}\sum_{j=i-\check{m}_n}^{i} \check{X}_{i-k}\mathcal{P}^j\check{X}_i - \sum_{j=1}^{l_n}\sum_{i=j}^{j+\check{m}_n} \check{X}_{i-k}\mathcal{P}^j\check{X}_i \right\| \\
&\le \left\| \sum_{j=1-\check{m}_n}^{0}\sum_{i=1}^{j+\check{m}_n} \check{X}_{i-k}\mathcal{P}^j\check{X}_i \right\| + \left\| \sum_{j=l_n-\check{m}_n+1}^{l_n}\sum_{i=l_n+1}^{j+\check{m}_n} \check{X}_{i-k}\mathcal{P}^j\check{X}_i \right\| \\
&\le 2\left(\sum_{j=1}^{\check{m}_n} \kappa_2^2\Theta_2(j)^2\right)^{1/2} \le 2\kappa_2\Theta_2\sqrt{\check{m}_n}
\end{aligned} \tag{S.29}$$

According to the proof of Theorem 2 of Wu (2009), when $k > 3\check{m}_n$ $\|M_{n,k}/\sqrt{n}\|^2 = \sum_{k\in\mathbb{Z}} \check{\gamma}_k^2$, where $\check{\gamma}_k = \mathbb{E}\check{X}_0\check{X}_k$. By (S.6) and (S.9), $|\check{\gamma}_k| \le \zeta_k$; and hence

$$\begin{aligned}
\|M_{n,k}/\sqrt{n}\|^2 \le \sum_{k\in\mathbb{Z}} \zeta_k^2 &= \sum_{j,j'=0}^{\infty}\left(\delta_2(j)\delta_2(j')\sum_{k\in\mathbb{Z}}\delta_2(j+k)\delta_2(j'+k)\right) \\
&\le \sum_{j,j'=0}^{\infty} \delta_2(j)\delta_2(j')\Psi_2^2 \le \Theta_2^2\Psi_2^2.
\end{aligned} \tag{S.30}$$

By (S.7) and (S.9), $\|\check{S}_{n,k}/\sqrt{l_n}\| \le 2\kappa_4\Theta_4$ for any $k \ge 0$. Combining (S.29) and (S.30), we have

$$\left|\mathbb{E}(\check{S}_{n,k_n}\check{S}_{n,k_n+h}) - \mathbb{E}(M_{n,k_n}M_{n,k_n+h})\right| \le (2\kappa_4\Theta_4 + \Theta_2\Psi_2)\sqrt{l_n} \cdot 2\kappa_2\Theta_2\sqrt{\check{m}_n}. \tag{S.31}$$

Observe that when $k_n > 3\check{m}_n$, $X_{q-k_n}X_{q'-k_n-h}$ and $\mathcal{P}^0X_q\mathcal{P}^0X_{q'}$ are independent for

$0 \le q, q' \le \check{m}_n$. Therefore,

$$\begin{aligned}
\mathbb{E}(M_{n,k_n} M_{n,k_n+h}) &= l_n \mathbb{E}\left(\sum_{q,q'=0}^{\check{m}_n} X_{q-k_n} X_{q'-k_n-h} \mathcal{P}^0 \check{X}_q \mathcal{P}^0 \check{X}_{q'} \right) \\
&= l_n \sum_{q,q'=0}^{\check{m}_n} \check{\gamma}_{q-q'+h} \mathbb{E}\left[(\mathcal{P}^0 \check{X}_q)(\mathcal{P}^0 \check{X}_{q'}) \right] \\
&= l_n \sum_{k\in\mathbb{Z}} \check{\gamma}_{k+h} \sum_{q'\in\mathbb{Z}} \mathbb{E}\left[(\mathcal{P}^0 \check{X}_{q'+k})(\mathcal{P}^0 \check{X}_{q'}) \right] \\
&= l_n \sum_{k\in\mathbb{Z}} \check{\gamma}_{k+h} \sum_{q'\in\mathbb{Z}} \mathbb{E}\left[(\mathcal{P}^{q'} \check{X}_k)(\mathcal{P}^{q'} \check{X}_0) \right] \\
&= l_n \sum_{k\in\mathbb{Z}} \check{\gamma}_{k+h} \check{\gamma}_k.
\end{aligned}$$

By (S.10), $|\gamma_k - \check{\gamma}_k| \le 2\kappa_2 \Psi_2(m+1)$. Since $|\gamma_k| \le \zeta_k$ and $|\check{\gamma}_k| \le \zeta_k$, we have

$$\begin{aligned}
\left| \sigma_h - \sum_{k\in\mathbb{Z}} \check{\gamma}_{k+h} \check{\gamma}_k \right| &= \left| \sum_{k\in\mathbb{Z}} (\gamma_k \gamma_{k+h} - \check{\gamma}_k \check{\gamma}_{k+h}) \right| \\
&\le 4\kappa_2 \Psi_2(m+1) \sum_{k\in\mathbb{Z}} \zeta_k \le 4\kappa_2 \Psi_2(m+1) \Theta_2^2. \qquad \text{(S.33)}
\end{aligned}$$

Combining (S.28), (S.31) and (S.33), the lemma follows by noting that κ_2, κ_4 are dominated by Θ_4; and $\Theta_2(\cdot)$, $\Psi_2(\cdot)$ and $\Psi_4(\cdot)$ are all dominated by $\Theta_4(\cdot)$. □

We now give the proof of Lemma 12.

Proof of Lemma 12. For $1 \le j \le w_n$, by (S.27), we have

$$\begin{aligned}
\left| \mathbb{E}(\bar{U}_{k,j} \bar{U}_{k+h,j}) - \mathbb{E}(U_{k,j} U_{k+h,j}) \right| &\le \|\bar{U}_{k,j} - U_{k,j}\| \|\bar{U}_{k+h,j}\| + \|U_{k,j}\| \|\bar{U}_{k+h,j} - U_{k+h,j}\| \\
&\le 4\kappa_4 \Theta_4 |H_j|^{1/2} \frac{(2\mathcal{C}_{p/2} \kappa_p \Theta_p)^{p/4} |H_j|^{p/8} (\log s_n)^{3(p-4)/4}}{n^{(p-4)/8}} \\
&\le C_p |H_j| n^{-(1-\gamma)(p-4)/8} (\log n)^{3(p-4)/4}.
\end{aligned}$$

Let $S_{k,j} = \sum_{i \in H_j} (X_{i-k}X_i - \gamma_k)$, by (S.7) and (S.13), we have

$$
\begin{aligned}
|\mathbb{E}(S_{k,j}S_{k+h,j}) - \mathbb{E}(U_{k,j}U_{k+h,j})| &\le \|S_{k,j} - U_{k,j}\| \|S_{k+h,j}\| + \|U_{k,j}\| \|S_{k+h,j} - U_{k+h,j}\| \\
&\le 24\kappa_4 \Theta_4^2 |H_j|^{1/2} \Theta_4(m_n - k + 1)|H_j|^{1/2} \le C|H_j| n^{-\alpha\beta}.
\end{aligned}
$$

Since $\Theta_4(m) = O(m^{-\alpha})$, elementary calculation shows that $\Delta_4(m) = O(n^{-\alpha^2/(1+\alpha)})$, which together with Lemma S.2 implies that if $k > t_n$,

$$
\begin{aligned}
|\mathbb{E}(U_{k,j}U_{k+h,j})/|H_j| - \sigma_h| &\le \Theta_4^3 \left(16\Delta_4(t_n/3+1) + 6\Theta_4\sqrt{t_n/l_n} + 4\Psi_4(t_n/3+1)\right) \\
&\le C\left(s_n^{-\alpha^2\iota/(1+\alpha)} + n^{-(1-\iota)\gamma/2}\right).
\end{aligned}
$$

Choose ℓ such that $0 < \ell < \min\{(1-\eta)(p-4)/8,\, \alpha\beta,\, \alpha^2\iota/(1+\alpha),\, (1-\iota)\gamma/2,\, \gamma - \beta\}$. Then

$$
\begin{aligned}
|\operatorname{Cov}(\mathcal{R}_{n,k}, \mathcal{R}_{n,k+h})/n - \sigma_h| \le C_p &\Big(n^{-(1-\eta)(p-4)/8}(\log n)^{(p-4)/4} + n^{-\alpha\beta} \\
&+ s_n^{-\alpha^2\iota/(1+\alpha)} + n^{-(1-\iota)\gamma/2}\Big) + \frac{2w_n m_n \sigma_0}{n} \le C_p\, s_n^{-\ell}
\end{aligned}
$$

and the lemma follows. □

S2.5 Step 5: Moderate deviations.

Proof of Lemma 13. Note that for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $|x+y|_\bullet \le |x|_\bullet + |y|$. Let $\boldsymbol{Z} \sim \mathcal{N}(0, I_d)$ and $\theta_n = (\log s_n)^{-1}$. Since $|\bar{U}_{k,j}| \le 2\sqrt{n}/(\log s_n)^3$, by Fact 2.2 of Einmahl and Mason (1997),

$$
\begin{aligned}
P(|\boldsymbol{\mathcal{R}}_n/\sqrt{n}|_\bullet \ge z_n) &\le P(|\Sigma_n^{1/2}\boldsymbol{Z}|_\bullet \ge z_n - \theta_n) + P(|\boldsymbol{\mathcal{R}}_n/\sqrt{n} - \Sigma_n^{1/2}\boldsymbol{Z}| \ge \theta_n) \\
&\le P(|\Sigma_n^{1/2}\boldsymbol{Z}|_\bullet \ge z_n - \theta_n) + C_{p,d}\exp\left\{-C_{p,d}^{-1}(\log s_n)^2\right\}.
\end{aligned}
$$

By Lemma S.8, the smallest eigenvalue of Σ is bounded from below by some $c_d > 0$ uniformly on $1 \le k_1 < k_2 < \cdots < k_d$. By Lemma 12 we have $\rho(\Sigma_n^{1/2} - \Sigma^{1/2}) \le c_d^{-1/2} \cdot \rho(\Sigma_n - \Sigma) \le C_{p,d}\, s_n^{-\ell}$, where the first inequality is taken from Problem 7.2.17 of Horn and Johnson (1990). It follows that

$$\begin{aligned} P(|\Sigma_n^{1/2}\boldsymbol{Z}|_\bullet \ge z_n - \theta_n) &\le P(|\Sigma^{1/2}\boldsymbol{Z}|_\bullet \ge z_n - 2\theta_n) + P\left[\left|\left(\Sigma_n^{1/2} - \Sigma^{1/2}\right)\boldsymbol{Z}\right| \ge \theta_n\right] \\ &\le P(|\Sigma^{1/2}\boldsymbol{Z}|_\bullet \ge z_n - 2\theta_n) + C_{p,d}\,\exp\left\{-C_{p,d}^{-1} s_n^{\ell}\right\}. \end{aligned}$$

By Lemma S.7, we have

$$P(|\Sigma^{1/2}\boldsymbol{Z}|_\bullet \ge z_n - 2\theta_n) \le \left[1 + C_{p,d}(\log s_n)^{-1/2}\right] P(|\Sigma^{1/2}\boldsymbol{Z}|_\bullet \ge z_n).$$

Putting these pieces together and observing that $\boldsymbol{V}$ and $\Sigma^{1/2}\boldsymbol{Z}$ have the same distribution, we have

$$P(|\boldsymbol{\mathcal{R}}_n/\sqrt{n}|_\bullet \ge z_n) \le \left[1 + C_{p,d}(\log s_n)^{-1/2}\right] P(|\boldsymbol{V}|_\bullet \ge z_n) + C_{p,d}\,\exp\left\{-C_{p,d}^{-1}(\log s_n)^2\right\},$$

which together with a similar lower bound completes the proof of Lemma 13. □

S2.6 Proof of Theorem 2

Proof of Theorem 2. We start with an m-dependence approximation that is similar to the proof of Theorem 1. Set $m_n = \lfloor n^\beta \rfloor$ for some $0 < \beta < 1$. Define $\tilde{X}_i = \mathcal{H}_{i-m_n} X_i$, $\tilde{\gamma}_k = \mathbb{E}(\tilde{X}_0 \tilde{X}_k)$, and $\tilde{R}_{n,k} = \sum_{i=k+1}^n (\tilde{X}_{i-k}\tilde{X}_i - \tilde{\gamma}_k)$. Similarly as the proof of Lemma 9, we have under the condition (14),

$$\max_{1 \le k < n} |R_{n,k} - \tilde{R}_{n,k}| = o_P\left(\sqrt{n/\log n}\right).$$

For $\tilde{R}_{n,k}$, we consider two cases according to whether $k \ge 3m_n$ or not.

Case 1: $k \geq 3m_n$. We first split the interval $[k+1, n]$ into the following big blocks of size $(k - m_n)$

$$H_j = [k + j - 1(k - m_n) + 1, k + j(k - m_n)] \quad \text{for } 1 \leq j \leq w_n - 1$$

$$H_{w_n} = [k + (w_n - 1)(k - m_n) + 1, n],$$

where w_n is the smallest integer such that $k + w_n(k - m_n) \geq n$. For each block H_j, we further split it into small blocks of size $2m_n$

$$K_{j,l} = [k + (j-1)(k - m_n) + (l-1)2m_n + 1, k + (j-1)(k - m_n) + 2lm_n] \quad \text{for } 1 \leq l < v_j$$

$$K_{j,v_j} = [k + (v_j - 1)(k - m_n) + (l-1)2m_n + 1, k + (j-1)(k - m_n) + |H_j|]$$

where v_j is the smallest integer such that $2m_n v_j \geq |H_j|$. Now define $U_{k,j,l} = \sum_{i \in K_{j,l}} \tilde{X}_{i-k}\tilde{X}_i$ and

$$\tilde{R}_{n,k}^{u,1} = \sum_{j \equiv u \pmod 3} \sum_{l \text{ odd}} U_{k,j,l} \quad \text{and} \quad \tilde{R}_{n,k}^{u,2} = \sum_{j \equiv u \pmod 3} \sum_{l \text{ even}} U_{k,j,l} \tag{S.34}$$

for $u = 0, 1, 2$. Observe that each $\tilde{R}_{n,k}^{u,o}$ $(u = 0, 1, 2;\ o = 1, 2)$ is a sum of independent random variables. By (S.7), $\|U_{k,j,l}\| \leq 2\kappa_4\Theta_4|U_{k,j,l}|^{1/2}$. By Corollary 1.7 of Nagaev (1979) where we take $y_i = \sqrt{n}$ in their result, we have for any $\lambda > 0$

$$\begin{aligned}
P\left(|\tilde{R}_{n,k}| \geq 6\lambda\sqrt{n \log n}\right) &\leq \sum_{u=0}^{2} \sum_{o=1,2} P\left(\left|\tilde{R}_{n,k}^{u,o}\right| \geq \lambda\sqrt{n \log n}\right) \\
&\leq \sum_{u=0}^{2} \sum_{o=1,2} \sum_{j,l}^{*} P\left(|U_{k,j,l}| \geq \lambda\sqrt{n \log n}\right) + 12\left(\frac{C_p\, n^{1-\beta} \cdot n^{\beta p/4}}{n^{p/4}}\right)^{p\sqrt{\log n}/(p+4)} \\
&\quad + 12 \exp\left\{-\frac{2\lambda^2}{(p+4)^2 \cdot e^{p/2} \cdot \kappa_4^2 \cdot \Theta_4^2} \cdot \log n\right\} =: I_{n,k} + II_{n,k} + III_{n,k},
\end{aligned} \tag{S.35}$$

where the range of j, l in the sum $\sum_{j,l}^{*}$ is as in (S.34). Clearly, $\sum_{k=3m_n}^{n-1} II_{n,k} = o(1)$. Similarly as the proof of Lemma 11, we can show that $\sum_{k=3m_n}^{n-1} I_{n,k} = o(1)$. Therefore, if $\epsilon = c_p/6$, then $\sum_{k=3m_n}^{n-1} III_{n,k} = O(n^{-1})$.

Case 2: $1 \le k < 3m_n$. This case is easier. By splitting the interval $[k+1, n]$ into blocks with size $4m_n$ and using a similar argument as (S.35), we have

$$\lim_{n\to\infty} \sum_{k=1}^{3m_n-1} P\left(|\tilde{R}_{n,k}| \ge c_p\sqrt{n\log n}\right) = 0.$$

The proof is complete. □

S3 Complements of Section 4.2

We prove the two intermediate steps in Section S3.1 and Section S3.2, Theorem 6 in Section S3.3, and Corollary 5 and 7 in Section S3.4.

S3.1 Step 2: Throw out small blocks.

To prove Lemma 15, we present an upper bound of $\mathrm{Cov}(R_{n,k}, R_{n,h})$ in Lemma S.4. We formulate the result in a more general way for later uses.

Let $\mathcal{A}_2$ be the collection of all double arrays $A = (a_{ij})_{i,j\ge1}$ such that

$$\|A\|_\infty := \max\left\{\sup_{i\ge1}\sum_{j=1}^\infty |a_{ij}|, \sup_{j\ge1}\sum_{i=1}^\infty |a_{ij}|\right\} < \infty.$$

Recall the definition of $\mathcal{A}_2$ in Section S3.1. For $A, B \in \mathcal{A}_2$, define $AB = (\sum_{k=1}^\infty a_{ik}b_{kj})$. It is easily seen that $AB \in \mathcal{A}_2$ and $\|AB\|_\infty \le \|A\|_\infty\|B\|_\infty$. Furthermore, this fact implies the following proposition, which will be useful in computing sums of products of cumulants. For $d \ge 0$, let $\mathcal{A}_d$ be the collection of all d-dimensional array $A = A(i_1, i_2, \ldots, i_d)$ such that

$$\|A\|_\infty := \max_{1\le j\le d}\left\{\sup_{i_j\ge1}\sum_{\{i_k:\, k\ne j\}} |A(i_1, i_2, \ldots, i_d)|\right\} < \infty.$$

Proposition S.3. *For $k \geq 0$, $l \geq 0$ and $d \geq 1$, if $A \in \mathcal{A}_{k+d}$ and $B \in \mathcal{A}_{l+d}$, define an array C by*

$$C(i_1, \ldots, i_k, i_{k+1}, \ldots, i_{k+l}) = \sum_{j_1, \ldots, j_d \geq 1} A(i_1, \ldots, i_k, j_1, \ldots, j_d) B(j_1, \ldots, j_d, i_{k+1}, \ldots, i_{k+l})$$

then $C \in \mathcal{A}_{k+l}$, and $\|C\|_\infty \leq \|A\|_\infty \|B\|_\infty$.

For a k-dimensional random vector $(Y_1, \ldots, Y_k)$ such that $\|Y_i\|_k < \infty$ for $1 \leq i \leq k$, denote by $\mathrm{Cum}(Y_1, \ldots, Y_k)$ its k-th order joint cumulant. For the stationary process $\{X_i\}_{i \in \mathbb{Z}}$, we write

$$\gamma(k_1, k_2, \ldots, k_d) := \mathrm{Cum}(X_0, X_{k_1}, X_{k_2}, \ldots, X_{k_d}).$$

Lemma S.4. *Assume $X_i \in \mathcal{L}^4$, $\mathbb{E}X_i = 0$, $\Theta_2 < \infty$ and $\sum_{k_1, k_2, k_3 \in \mathbb{Z}} |\gamma(k_1, k_2, k_3)| < \infty$. For $k, h \geq 1$, $l_n \geq t_n > 0$ and $s_n \in \mathbb{Z}$, set $U_k = \sum_{i=1}^{l_n} (X_{i-k} X_i - \gamma_k)$ and $V_h = \sum_{j=s_n+1}^{s_n+t_n} (X_{j-h} X_j - \gamma_j)$, then we have*

$$|\mathbb{E}(U_k V_h)| \leq t_n \Xi(k, h)$$

where $[\Xi(k, h)_{k,h \geq 1}]$ is a symmetric double array of non-negative numbers such that $\Xi \in \mathcal{A}_2$, and

$$\|\Xi\|_\infty \leq 2\Theta_2^4 + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} |\gamma(k_1, k_2, k_3)|.$$

Remark S.2. In Lemma S.4, as well as in the proofs of Lemma 15 and Lemma 16, we need the summability of joint cumulants. For this reason, we provide a sufficient condition in Theorem S.6.

In the proof of Lemma 15, we need the concept of *indecomposable partitions*. Consider the table

$$\begin{matrix} (1,1) & \dots & (1,J_1) \\ \vdots & & \vdots \\ (I,1) & \dots & (I,J_I) \end{matrix} .$$

Denote the j-th row of the table by ϑ_j. A partition $\boldsymbol{\nu} = \{\nu_1, \dots, \nu_q\}$ of the table is said to be *indecomposable* if there are no sets $\nu_{i_1}, \dots, \nu_{i_k}$ $(k < q)$ and rows $\vartheta_{j_1}, \dots, \vartheta_{j_l}$ $(l < I)$ such that $\nu_{i_1} \cup \dots \cup \nu_{i_k} = \vartheta_{j_1} \cup \dots \cup \vartheta_{j_l}$.

Proof of Lemma 15. Write

$$\begin{aligned}\sum_{k=1}^{s_n} \mathbb{E}_0(\tilde{R}_{n,k}^2 - \mathcal{R}_{n,k}^2) &= 2\sum_{k=1}^{s_n} \mathbb{E}_0\left[\mathcal{R}_{n,k}(\tilde{R}_{n,k} - \mathcal{R}_{n,k})\right] + \sum_{k=1}^{s_n} \mathbb{E}_0(\tilde{R}_{n,k} - \mathcal{R}_{n,k})^2 \\ &=: 2I_n + II_n.\end{aligned}$$

Using Lemma 16, we know $II_n/(n\sqrt{s_n}) = o_P(1)$. We can express I_n as

$$I_n = \sum_{a=0}^{1}\sum_{b=0}^{1} I_{n,ab} = I_{n,00} + I_{n,01} + I_{n,10} + I_{n,11}. \tag{S.36}$$

where for $a, b = 0, 1$ (assume without loss of generality that w_n is even),

$$I_{n,ab} = \sum_{k=1}^{s_n} \mathbb{E}_0\left(\sum_{j=0}^{w_n/2} U_{k,2j-a} \sum_{j=0}^{w_n/2} V_{k,2j-b}\right).$$

Consider the first term in (S.36), write

$$\begin{aligned}\mathbb{E}(I_{n,00}^2) = &\sum_{k,h=1}^{s_n} \mathbb{E}\left[\sum_{j=1}^{w_n/2} \mathbb{E}_0(U_{k,2j}V_{k,2j}) \cdot \mathbb{E}_0(U_{h,2j}V_{h,2j})\right] \\ &+ \sum_{k,h=1}^{s_n} \sum_{j_1 \neq j_2} \mathbb{E}(U_{k,2j_1}U_{h,2j_1})\mathbb{E}(V_{k,2j_2}V_{h,2j_2}) \\ &+ \sum_{k,h=1}^{s_n} \sum_{j_1 \neq j_2} \mathbb{E}(U_{k,2j_1}V_{h,2j_1})\mathbb{E}(V_{k,2j_2}U_{h,2j_2}) \\ := &A_n + B_n + C_n.\end{aligned}$$

By Lemma S.4, it holds that

$$\begin{aligned}|B_n| &\leq \sum_{k,h=1}^{s_n}\sum_{j_1,j_2=0}^{w_n/2} l_n|K_{2j_2}|\cdot\left[\tilde{\Xi}(k,h)\right]^2\\ &\leq w_n l_n\cdot(w_n m_n+2l_n)\sum_{k,h=1}^{s_n}\left[\tilde{\Xi}_n(k,h)\right]^2 = o(n^2 s_n),\end{aligned}$$

where $\tilde{\Xi}_n(k,h)$ is the $\Xi(k,h)$ (defined in Lemma S.4) for the sequence $(\tilde{X}_i)$. Similarly,

$$\begin{aligned}|C_n| &\leq \sum_{k,h=1}^{s_n}\sum_{j_1,j_2=1}^{w_n/2} |K_{2j_1}|\cdot|K_{2j_2}|\cdot\left[\tilde{\Xi}_n(k,h)\right]^2\\ &\leq (w_n m_n+l_n)^2\sum_{k,h=1}^{s_n}\left[\tilde{\Xi}_n(k,h)\right]^2 = o(n^2 s_n).\end{aligned}$$

To deal with A_n, we express it in terms of cumulants

$$\begin{aligned}A_n &= \sum_{k,h=1}^{s_n}\sum_{j=1}^{w_n/2}[\mathrm{Cum}(U_{k,2j},V_{k,2j},U_{h,2j},V_{h,2j})\\ &\qquad +\mathbb{E}(U_{k,2j}U_{h,2j})\mathbb{E}(V_{k,2j}V_{h,2j})\\ &\qquad +\mathbb{E}(U_{k,2j}V_{h,2j})\mathbb{E}(V_{k,2j}U_{h,2j})]\\ &=: D_n+E_n+F_n.\end{aligned}$$

Apparently $|E_n| = o(n^2 s_n)$ and $|F_n| = o(n^2 s_n)$. Using the multilinearity of cumulants, we have

$$\begin{aligned}&\mathrm{Cum}(U_{k,2j},V_{k,2j},U_{h,2j},V_{h,2j})\\ &= \sum_{i_1,i_2\in H_{2j}}\sum_{j_1,j_2\in K_{2j}}\mathrm{Cum}(\tilde{X}_{i_1-k}\tilde{X}_{i_1},\tilde{X}_{j_1-k}\tilde{X}_{j_1},\tilde{X}_{i_2-h}\tilde{X}_{i_2},\tilde{X}_{j_2-h}\tilde{X}_{j_2})\end{aligned}$$

for $1\leq k,h\leq s_n$. By Theorem II.2 of Rosenblatt (1985), we know

$$\mathrm{Cum}\left(\tilde{X}_{i_1-k}\tilde{X}_{i_1},\tilde{X}_{j_1-k}\tilde{X}_{j_1},\tilde{X}_{i_2-h}\tilde{X}_{i_2},\tilde{X}_{j_2-h}\tilde{X}_{j_2}\right) = \sum_{\boldsymbol{\nu}}\prod_{q=1}^{b}\mathrm{Cum}(\tilde{X}_i,\ i\in\nu_q) \quad \text{(S.37)}$$

where the sum is over all indecomposable partitions $\boldsymbol{\nu} = \{\nu_1,\ldots,\nu_q\}$ of the table

$$\begin{array}{cc} i_1 - k & i_1 \\ j_1 - k & j_1 \\ i_2 - h & i_2 \\ j_2 - h & j_2 \end{array}$$

By Theorem S.6, the condition $\sum_{k=0}^{\infty} k^6 \delta_8(k) < \infty$ implies that all the joint cumulants up to order eight are absolutely summable. Therefore, using Proposition S.3, we know

$$\sum_{k,h=1}^{s_n} |\mathrm{Cum}(U_{k,2j}, V_{k,2j}, U_{h,2j}, V_{h,2j})| = O(|K_{2j}|s_n^2),$$

and it follows that $|D_n| = O\left((w_n m_n + l_n)s_n^2\right) = o(n^2 s_n)$. We have shown that $\mathbb{E}(I_{n,00}^2) = o(n^2 s_n)$, which, in conjunction with similar results for the other three terms in (S.36), implies that $\mathbb{E}(I_n^2) = o(n^2 s_n)$ and hence $I_n/(n\sqrt{s_n}) = o_P(1)$. The proof is now complete.

$\square$

It remains to prove Lemma S.4.

Proof of Lemma S.4. Write

$$\begin{aligned} \mathbb{E}(U_k V_h) &= \sum_{i=1}^{l_n} \sum_{j=1}^{t_n} \mathbb{E}[(X_{i-k}X_i - \gamma_k)(X_{s_n+j-h}X_{s_n+j} - \gamma_h)] \\ &= \sum_{i=1}^{l_n} \sum_{j=1}^{t_n} [\gamma(-k, j+s_n-i-h, j+s_n-i) \\ &\qquad + \gamma_{j+s_n-i+k-h}\gamma_{j+s_n-i} + \gamma_{j+s_n-i+k}\gamma_{j+s_n-i-h}]. \end{aligned}$$

For the sum of the second term, we have

$$\left| \sum_{i=1}^{l_n} \sum_{j=1}^{t_n} \gamma_{j+s_n-i+k-h}\gamma_{j+s_n-i} \right| = \left| \sum_{d=1}^{t_n-1} (\gamma_{s_n+d+k-h}\gamma_{s_n+d})(t_n - d) \right.$$

$$
\begin{aligned}
&+ t_n \sum_{d=t_n-l_n}^{0} \gamma_{s_n+d+k-h}\gamma_{s_n+d} \\
&+ \sum_{d=1-l_n}^{t_n-l_n-1} (\gamma_{s_n+d+k-h}\gamma_{s_n+d})(l_n+d)\Bigg| \\
\le & t_n \sum_{d\in\mathbb{Z}} |\gamma_{s_n+d+k-h}\gamma_{s_n+d}| \\
\le & t_n \sum_{d\in\mathbb{Z}} \zeta_{d+k-h}\zeta_d.
\end{aligned}
$$

Similarly, for the sum of the last term

$$
\left|\sum_{i=1}^{l_n}\sum_{j=1}^{t_n} \gamma_{j+s_n-i+k}\gamma_{j+s_n-i-h}\right| \le t_n \sum_{d\in\mathbb{Z}} \zeta_{d+k+h}\zeta_d.
$$

Observe that $\sum_{h=1}^{\infty}\sum_{d\in\mathbb{Z}} \zeta_{d+k-h}\zeta_d \le \left(\sum_{d\in\mathbb{Z}}\zeta_d\right)^2 \le \Theta_2^4$ and similarly $\sum_{h=1}^{\infty}\sum_{d\in\mathbb{Z}} \zeta_{d+k+h}\zeta_d \le \Theta_2^4$. For the sum of the first term, it holds that

$$
\left|\sum_{i=1}^{l_n}\sum_{j=1}^{t_n} \gamma(-k, j+s_n-i-h, j+s_n-i)\right| \le t_n \sum_{d\in\mathbb{Z}} |\gamma(-k, d-h, d)|.
$$

Utilizing the summability of cumulants, the proof is complete. □

S3.2 Step 3: Central limit theorem concerning $\mathcal{R}_{n,k}$'s.

Proof of Lemma 16. Let $\Upsilon_n(k,h) := \mathbb{E}(U_{k,1}U_{h,1})$ and $\upsilon_n(k,h) := \Upsilon_n(k,h)/l_n$. By Lemma S.4 we know $|\upsilon_n(k,h)| \le \tilde{\Xi}_n(k,h)$. Write

$$
\begin{aligned}
\sum_{k=1}^{s_n} \mathbb{E}_0 \mathcal{R}_{n,k}^2 &= \sum_{k=1}^{s_n}\left[\sum_{j=1}^{w_n}\left(U_{k,j}^2 - \Upsilon_n(k,k)\right) + 2\sum_{j=1}^{w_n}\left(U_{k,j}\sum_{l=1}^{j-1}U_{k,l}\right)\right] \\
&= \sum_{j=1}^{w_n}\left[\sum_{k=1}^{s_n}\left(U_{k,j}^2 - \Upsilon_n(k,k)\right)\right] + 2\sum_{j=1}^{w_n}\left(\sum_{k=1}^{s_n}U_{k,j}\sum_{l=1}^{j-1}U_{k,l}\right).
\end{aligned}
$$

Using similar a argument as the one for dealing with the term A_n in Lemma 15, we know

$$
\sum_{j=1}^{w_n}\left\|\sum_{k=1}^{s_n}\left(U_{k,j}^2 - \Upsilon_n(k,k)\right)\right\|^2 = o(n^2 s_n),
$$

and it follows that

$$\frac{1}{n\sqrt{s_n}}\sum_{j=1}^{w_n}\left[\sum_{k=1}^{s_n}\left(U_{k,j}^2-\Upsilon_n(k,k)\right)\right]=o_P(1).$$

Therefore, it suffices to consider

$$\sum_{j=1}^{w_n}\left(\sum_{k=1}^{s_n}U_{k,j}\sum_{l=1}^{j-1}U_{k,l}\right)=:\sum_{j=1}^{w_n}D_{n,j}.$$

Let $\mathcal{G}_{n,j}=\langle D_{n,1},\ldots,D_{n,j}\rangle$. Observe that $(D_{n,j})$ is a martingale difference sequence with respect to $(\mathcal{G}_{n,j})$. We shall apply the martingale central limit theorem. Write

$$\begin{aligned}\mathbb{E}\left(D_{n,j}^2|\mathcal{G}_{n,j-1}\right)-\mathbb{E}D_{n,j}^2 &= \sum_{k,h=1}^{s_n}\Upsilon_n(k,h)\left(\sum_{l=1}^{j-1}U_{k,l}\sum_{l=1}^{j-1}U_{h,l}-(j-1)\Upsilon_n(k,h)\right)\\ &=\sum_{k,h=1}^{s_n}\Upsilon_n(k,h)\left(\sum_{l=1}^{j-1}U_{k,l}U_{h,l}-(j-1)\Upsilon_n(k,h)\right)\\ &+\sum_{k,h=1}^{s_n}\Upsilon_n(k,h)\left(\sum_{l=1}^{j-1}U_{k,l}\sum_{q=1}^{l-1}U_{h,q}+\sum_{l=1}^{j-1}U_{h,l}\sum_{q=1}^{l-1}U_{k,q}\right)\\ &=: I_{n,j}+II_{n,j}\end{aligned}$$

For the first term, by Lemma S.4, we have

$$\begin{aligned}\left\|\sum_{j=1}^{w_n}I_{n,j}\right\|^2 &= \left\|\sum_{j=1}^{w_n-1}(w_n-j)\sum_{k,h=1}^{s_n}\Upsilon_n(k,h)\left[U_{k,j}U_{h,j}-\Upsilon_n(k,h)\right]\right\|^2\\ &=\sum_{j=1}^{w_n-1}(w_n-j)^2\left[\sum_{k,h}|\Upsilon_n(k,h)|\,\|(U_{k,j}U_{h,j}-\Upsilon_n(k,h))\|\right]^2\\ &\le w_n^3l_n^4\left[\sum_{k,h}|\upsilon_n(k,h)|\cdot 4\Theta_8^2\right]^2=o(n^4s_n^2).\end{aligned}$$

Using Lemma S.4 and Proposition S.3, we obtain

$$\left\|\sum_{j=1}^{w_n}II_{n,j}\right\|^2=\left\|\sum_{j=1}^{w_n-1}(w_n-j)\sum_{k,h}\Upsilon_n(k,h)\left(U_{k,j}\sum_{l=1}^{j-1}U_{h,l}+U_{h,j}\sum_{l=1}^{j-1}U_{k,l}\right)\right\|^2$$

$$
\begin{aligned}
=&2\sum_{j=1}^{w_n-1}(w_n-j)^2(j-1)\Bigg\{\sum_{1\le k_1,h_1,k_2,h_2\le s_n}\Upsilon_n(k_1,h_1)\Upsilon_n(k_2,h_2)\\
&\qquad\qquad\times\left[\Upsilon_n(k_1,k_2)\Upsilon_n(h_1,h_2)+\Upsilon_n(k_1,h_2)\Upsilon_n(h_1,k_2)\right]\Bigg\}\\
\le&4n^4\sum_{1\le k_1,h_1,k_2,h_2\le s_n}|\upsilon_n(k_1,h_1)\upsilon_n(h_1,h_2)\upsilon_n(h_2,k_2)\upsilon_n(k_2,k_1)|=O(n^4 s_n)=o(n^4 s_n^2).
\end{aligned}
$$

Therefore, we have

$$
\frac{1}{n^2 s_n}\left[\sum_{j=1}^{w_n}\mathbb{E}\left(D_{n,j}^2|\mathcal{G}_{n,j-1}\right)-\sum_{j=1}^{w_n}\mathbb{E}D_{n,j}^2\right]\xrightarrow{p}0.
$$

Using Lemma S.4 and Lemma S.2, we know

$$
\frac{1}{n^2 s_n}\sum_{j=1}^{w_n}\mathbb{E}D_{n,j}^2=\frac{1}{2n^2 s_n}w_n(w_n-1)l_n^2\sum_{k,h=1}^{s_n}[\upsilon_n(k,h)]^2\to\frac{1}{2}\sum_{k\in\mathbb{Z}}\sigma_k^2,
$$

and it follows that

$$
\frac{1}{n^2 s_n}\sum_{j=1}^{w_n}\mathbb{E}\left(D_{n,j}^2|\mathcal{G}_{n,j-1}\right)\xrightarrow{p}\frac{1}{2}\sum_{k\in\mathbb{Z}}\sigma_k^2. \tag{S.38}
$$

To verify the Lindeberg condition, we compute

$$
\begin{aligned}
\mathbb{E}D_{n,j}^4=&\sum_{k_1,k_2,k_3,k_4=1}^{s_n}\mathbb{E}\left(U_{k_1,j}U_{k_2,j}U_{k_3,j}U_{k_4,j}\right)\\
&\times\mathbb{E}\left[\left(\sum_{l=1}^{j-1}U_{k_1,l}\right)\left(\sum_{l=1}^{j-1}U_{k_2,l}\right)\left(\sum_{l=1}^{j-1}U_{k_3,l}\right)\left(\sum_{l=1}^{j-1}U_{k_4,l}\right)\right]\\
\le&\sum_{k_1,k_2,k_3,k_4=1}^{s_n}|\mathbb{E}(U_{k_1,j}U_{k_2,j}U_{k_3,j}U_{k_4,j})|\cdot 2\mathcal{C}_4^4(j-1)^2 l_n^2\Theta_8^8
\end{aligned}
$$

We express $\mathbb{E}(U_{k_1,1}U_{k_2,1}U_{k_3,1}U_{k_4,1})$ in terms of cumulants

$$
\begin{aligned}
&\mathbb{E}(U_{k_1,1}U_{k_2,1}U_{k_3,1}U_{k_4,1})\\
&=\mathrm{Cum}(U_{k_1,1},U_{k_2,1},U_{k_3,1},U_{k_4,1})+\mathbb{E}(U_{k_1,1}U_{k_2,1})\mathbb{E}(U_{k_3,1}U_{k_4,1})\\
&\quad+\mathbb{E}(U_{k_1,1}U_{k_3,1})\mathbb{E}(U_{k_2,1}U_{k_4,1})+\mathbb{E}(U_{k_1,1}U_{k_4,1})\mathbb{E}(U_{k_2,1}U_{k_3,1})\\
&=:A_n+B_n+E_n+F_n
\end{aligned}
$$

From Lemma S.4, it is easily seen that

$$\sum_{k_1,k_2,k_3,k_4=1}^{s_n} |B_n| \le l_n^2 \sum_{k_1,k_2,k_3,k_4=1}^{s_n} \tilde{\Xi}_n(k_1,k_2) \cdot \tilde{\Xi}_n(k_3,k_4) = O(l_n^2 s_n^2),$$

and similarly $\sum_{k_1,k_2,k_3,k_4=1}^{s_n} |E_n| = O(l_n^2 s_n^2)$ and $\sum_{k_1,k_2,k_3,k_4=1}^{s_n} |F_n| = O(l_n^2 s_n^2)$. By multilinearity of cumulants,

$$A_n = \sum_{i_1,i_2,i_3,i_4=1}^{l_n} \mathrm{Cum}(\tilde{X}_{i_1-k_1}\tilde{X}_{i_1}, \tilde{X}_{i_2-k_2}\tilde{X}_{i_2}, \tilde{X}_{i_3-k_3}\tilde{X}_{i_3}, \tilde{X}_{i_4-k_4}\tilde{X}_{i_4}).$$

Each cumulant in the preceding equation is to be further simplified similarly as (S.37). Using summability of joint cumulants up to order eight and Proposition S.3, we have

$$\sum_{k_1,k_2,k_3,k_4=1}^{s_n} |A_n| = O(l_n s_n^3) = o(l_n^2 s_n^2).$$

Using orders for $|A_n|$, $|B_n|$, $|E_n|$ and $|F_n|$, we obtain $\sum_{j=1}^{w_n} \mathbb{E}D_{n,j}^4 = o(n^4 s_n^2)$. Then, by (S.38), we can apply Corollary 3.1. of Hall and Heyde (1980) to obtain

$$\frac{1}{n\sqrt{s_n}} \sum_{j=1}^{w_n} D_{n,j} \Rightarrow \mathcal{N}\left(0, \frac{1}{2}\sum_{k\in\mathbb{Z}} \sigma_k^2\right),$$

and the lemma follows. □

S3.3 Proof of Theorem 6

Proof of Theorem 6. We shall only prove (22), since (21) can be obtained by very similar arguments. Write $\hat{\gamma}_k = \mathbb{E}_0\hat{\gamma}_k + \gamma_k - (\gamma_k - \mathbb{E}\hat{\gamma}_k)$, and hence

$$\begin{aligned}\sum_{k=1}^{s_n}(\hat{\gamma}_k^2 - \gamma_k^2) &= 2\sum_{k=1}^{s_n}\gamma_k\mathbb{E}_0\hat{\gamma}_k + \sum_{k=1}^{s_n}(\mathbb{E}_0\hat{\gamma}_k)^2 - 2\sum_{k=1}^{s_n}\frac{k}{n}\gamma_k\mathbb{E}_0\hat{\gamma}_k - 2\sum_{k=1}^{s_n}\frac{k}{n}\gamma_k^2 + \sum_{k=1}^{s_n}\frac{k^2}{n^2}\gamma_k^2 \\ &=: 2I_n + II_n + III_n + IV_n + V_n.\end{aligned}$$

Using the conditions $\Theta_4 < \infty$ and $s_n = o(\sqrt{n})$, it is easily seen that $\sqrt{n}IV_n \to 0$ and $\sqrt{n}V_n \to 0$. Furthermore

$$\sqrt{n}\|III_n\| \leq 2\sqrt{n}\sum_{k=1}^{s_n}\frac{k}{n}|\gamma_k| \cdot \frac{2\Theta_4^2}{\sqrt{n}} \to 0 \quad \text{and} \quad \sqrt{n}\mathbb{E}II_n \leq \sqrt{n}\sum_{k=1}^{s_n}\frac{4\Theta_4^4}{n} \to 0.$$

Define $Y_i = \sum_{k=1}^{\infty}\gamma_k X_{i-k}$. For the term I_n, write

$$\begin{aligned} nI_n &= \sum_{i=1}^{n}\mathbb{E}_0(X_iY_i) - \sum_{i=1}^{n}\mathbb{E}_0\left(X_i\sum_{k=s_n+1}^{\infty}\gamma_kX_{i-k}\right) + \sum_{k=1}^{s_n}\gamma_k\left(\sum_{i=1}^{k}(X_{i-k}X_i - \gamma_k)\right) \\ &=: A_n + B_n + E_n \end{aligned}$$

Clearly $\|E_n\|/\sqrt{n} \leq \sum_{k=1}^{s_n}|\gamma_k|2\Theta_4^2\sqrt{k}/\sqrt{n} \to 0$. Define $W_{n,i} = X_i\sum_{k=s_n+1}^{\infty}\gamma_kX_{i-k}$, then

$$\|\mathcal{P}^0W_{n,i}\| \leq \begin{cases} \delta_4(i)\cdot\Theta_4\sum_{k=s_n+1}^{\infty}|\gamma_k| & \text{if} \quad 0 \leq i \leq s_n \\ \Theta_4\delta_4(i)\sum_{k=s_n+1}^{\infty}|\gamma_k| + \Theta_4\sum_{k=s_n+1}^{i}|\gamma_k|\delta_4(i-k) & \text{if} \quad i > s_n. \end{cases}$$

It follows that

$$\|B_n/\sqrt{n}\| \leq 2\Theta_4^2\sum_{k=s_n+1}^{\infty}|\gamma_k| \to 0.$$

Set $Z_i = X_iY_i$, then (Z_i) is a stationary process of the form (9). Furthermore

$$\|\mathcal{P}^0Z_i\| \leq \delta_4(i)\cdot\Theta_4\sum_{k=1}^{\infty}|\gamma_k| + \Theta_4\sum_{k=1}^{i}|\gamma_k|\delta_4(i-k).$$

Since $\sum_{i=0}^{\infty}\|\mathcal{P}^0Z_i\| < \infty$, utilizing Theorem 1 in Hannan (1973) we have $A_n/\sqrt{n} \Rightarrow \mathcal{N}(0, \|D_0\|^2)$, and then (22) follows. □

S3.4 Proof of Corollary 5 and 7

Proof of Corollary 5 and 7. By (S.5), we know $\|n\bar{X}_n\|_4 \leq \sqrt{3n}\Theta_4$, and it follows that

$$\left\|\sum_{i=k+1}^{n}(X_{i-k} - \bar{X}_n)(X_i - \bar{X}_n) - \sum_{i=k+1}^{n}X_{i-k}X_i\right\| \leq 9\Theta_4^2.$$

Theorem 4 holds for $\breve{\gamma}_k$ because

$$\frac{n}{\sqrt{s_n}}\sum_{k=1}^{s_n}\mathbb{E}\left|(\hat{\gamma}_k-\mathbb{E}\hat{\gamma}_k)^2-(\breve{\gamma}_k-\mathbb{E}\hat{\gamma}_k)^2\right| \le \frac{n}{\sqrt{s_n}}\sum_{k=1}^{s_n}\|\hat{\gamma}_k+\breve{\gamma}_k-2\mathbb{E}\hat{\gamma}_k\|\cdot\|\hat{\gamma}_k-\breve{\gamma}_k\|$$
$$\le \frac{n}{\sqrt{s_n}}\sum_{k=1}^{s_n}\left(\frac{4\Theta_4^2}{\sqrt{n}}+\frac{9\Theta_4^2}{n}\right)\cdot\frac{9\Theta_4^2}{n}\to 0.$$

In Theorem 6, (22) holds with $\hat{\gamma}_k$ replaced by $\breve{\gamma}_k$ because

$$\sqrt{n}\sum_{k=1}^{s_n}\mathbb{E}\left|\hat{\gamma}_k^2-\breve{\gamma}_k^2\right| \le \sqrt{n}\sum_{k=1}^{s_n}\|\hat{\gamma}_k+\breve{\gamma}_k\|\cdot\|\hat{\gamma}_k-\breve{\gamma}_k\|$$
$$\le \sqrt{n}\sum_{k=1}^{s_n}\left(2|\gamma_k|+\frac{4\Theta_4^2}{\sqrt{n}}+\frac{9\Theta_4^2}{n}\right)\frac{9\Theta_4^2}{n}\to 0,$$

and (21) can be proved similarly. Now we turn to the sample autocorrelations. Write

$$\sum_{k=1}^{s_n}\left\{[\hat{r}_k-(1-k/n)r_k]^2-[\hat{\gamma}_k/\gamma_0-(1-k/n)r_k]^2\right\}$$
$$=\sum_{k=1}^{s_n}\frac{2(\mathbb{E}_0\hat{\gamma}_k)[\hat{\gamma}_k(\gamma_0-\hat{\gamma}_0)]}{\gamma_0^2\hat{\gamma}_0}+\frac{\hat{\gamma}_k^2(\gamma_0-\hat{\gamma}_0)^2}{\gamma_0^2\hat{\gamma}_0^2}.$$

Since

$$\sum_{k=1}^{s_n}\mathbb{E}\left|(\mathbb{E}_0\hat{\gamma}_k)\hat{\gamma}_k(\gamma_0-\hat{\gamma}_0)\right| \le \sum_{k=1}^{s_n}2\mathcal{C}_3\Theta_6^2\frac{1}{\sqrt{n}}\cdot\left(|\gamma_k|+2\mathcal{C}_3\Theta_6^2\frac{1}{\sqrt{n}}\right)\cdot 2\mathcal{C}_3\Theta_6^2\frac{1}{\sqrt{n}}=o\left(\frac{\sqrt{s_n}}{n}\right)$$

and similarly $\sum_{k=1}^{s_n}\mathbb{E}\left|\hat{\gamma}_k^2(\gamma_0-\hat{\gamma}_0)^2\right|=o(\sqrt{s_n}/n)$, (19) follows by applying the Slutsky theorem. To show the limit theorems in Corollary 7 note that using the Cramer-Wold device, we have

$$\left[\sqrt{n}(\hat{\gamma}_0^2-\gamma_0^2),\sqrt{n}\left(\sum_{k=1}^{s_n}\hat{\gamma}_k^2-\sum_{k=1}^{s_n}\gamma_k^2\right)\right]$$

converges to a bivariate normal distribution. Then Corollary 7 follows by applying the delta method. $\square$

S4 A Normal Comparison Principle

In this section we shall control tail probabilities of Gaussian vectors by using their covariance matrices. Denote by $\varphi_d((r_{ij}); x_1, \ldots, x_d)$ the density of a d-dimensional multivariate normal random vector $\boldsymbol{X} = (X_1, \ldots, X_d)^\top$ with mean zero and covariance matrix (r_{ij}), where we always assume $r_{ii} = 1$ for $1 \le i \le d$ and (r_{ij}) is nonsingular. For $1 \le h < l \le d$, we use $\varphi_2((r_{ij}); X_h = x_h, X_l = x_l)$ to denote the marginal density of the sub-vector $(X_h, X_l)^\top$. Let

$$Q_d\left((r_{ij}); z_1, \ldots, z_d\right) = \int_{z_1}^{\infty} \cdots \int_{z_d}^{\infty} \varphi_d\left((r_{ij}), x_1, \ldots, x_d\right) \, \mathrm{d}x_d \cdots \mathrm{d}x_1.$$

The partial derivative with respect to r_{hl} is obtained similarly as equation (3.6) of Berman (1964) by using equation (3) of Plackett (1954)

$$\begin{aligned} &\frac{\partial Q_d\left((r_{ij}); z_1, \ldots, z_d\right)}{\partial r_{hl}} \\ &= \left(\prod_{k \neq h,l} \int_{z_k}^{\infty}\right) \varphi_d\left((r_{ij}); x_1, \ldots, x_{h-1}, z_h, x_{h+1}, \ldots, x_{l-1}, z_l, x_{l+1}, \ldots, x_d\right) \prod_{k \neq h,l} \mathrm{d}x_k. \end{aligned} \tag{S.39}$$

where $\left(\prod_{k \neq h,l} \int_{z_k}^{\infty}\right)$ stands for $\int_{z_1}^{\infty} \cdots \int_{z_{h-1}}^{\infty} \int_{z_{h+1}}^{\infty} \cdots \int_{z_{l-1}}^{\infty} \int_{z_{l+1}}^{\infty} \cdots \int_{z_d}^{\infty}$. If all the z_k have the same value z, we use the simplified notation $Q_d\left((r_{ij}); z\right)$ and $\partial Q_d((r_{ij}); z)/\partial r_{hl}$. The following simple facts about conditional distribution will be useful. For four different indicies $1 \le h, l, k, m \le d$, we have

$$\mathbb{E}(X_k | X_h = X_l = z) = \frac{r_{kh} + r_{kl}}{1 + r_{hl}} z, \tag{S.40}$$

$$\operatorname{Var}(X_k | X_h = X_l = z) = \frac{1 - r_{hl}^2 - r_{kh}^2 - r_{kl}^2 + 2 r_{hl} r_{kh} r_{kl}}{1 - r_{hl}^2}, \tag{S.41}$$

$$\operatorname{Cov}(X_k, X_m | X_h = X_l = z) = r_{km} - \frac{r_{hk} r_{hm} + r_{lk} r_{lm} - r_{hl} r_{hk} r_{lm} - r_{hl} r_{hm} r_{lk}}{1 - r_{hl}^2}. \tag{S.42}$$

Lemma S.5. *For every $z > 0$, $0 < s < 1$, $d \geq 1$ and $\epsilon > 0$, there exists positive constants C_d and ϵ_d such that for $0 < \epsilon < \epsilon_d$*

1. *if $|r_{ij}| < \epsilon$ for all $1 \leq i < j \leq d$, then*

$$Q_d\left((r_{ij}); z\right) \leq C_d \exp\left\{-\left(\frac{d}{2} - C_d\epsilon\right) z^2\right\} \tag{S.43}$$

$$Q_d\left((r_{ij}); z, \ldots, z\right) \leq C_d\, f_d(\epsilon, 1/z)\, \exp\left\{-\left(\frac{d}{2} - C_d\epsilon\right) z^2\right\} \tag{S.44}$$

$$Q_d\left((r_{ij}); sz, z, \ldots, z\right) \leq C_d \exp\left\{-\left(\frac{s^2 + d - 1}{2} - C_d\epsilon\right) z^2\right\} \tag{S.45}$$

where $f_{2k}(x, y) = \sum_{l=0}^{k} x^l y^{2(k-l)}$ and $f_{2k-1}(x, y) = \sum_{l=0}^{k-1} x^l y^{2(k-l)-1}$ for $k \geq 1$;

2. *if for all $1 \leq i < j \leq d+1$ such that $(i, j) \neq (1, 2)$, $|r_{ij}| \leq \epsilon$, then*

$$Q_{d+1}\left((r_{ij}); z\right) \leq C_d \exp\left\{-\left(\frac{(1 - |r_{12}|)^2 + d}{2} - C_d\epsilon\right) z^2\right\}. \tag{S.46}$$

Proof. The following facts about normal tail probabilities are well-known:

$$P(X_1 \geq x) \leq \frac{1}{\sqrt{2\pi} x} e^{-x^2/2} \text{ for } x > 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{P(X_1 \geq x)}{(1/x)(2\pi)^{-1/2} \exp\{-x^2/2\}} = 1, \tag{S.47}$$

By (S.47), the inequalities (S.43) – (S.45) with $\epsilon = 0$ are true for the random vector with iid standard normal entries. The idea is to compare the desired probability with the corresponding one for such a vector. We first prove (S.43) by induction. When $d = 1$, the inequality is trivially true. When $d = 2$, by (S.39), there exists a number r'_{12} between 0 and r_{12} such that

$$\begin{aligned} |Q_2((r_{ij}); z) - Q_2(I_2; z)| &\leq \varphi((r'_{ij}), z, z)|r_{12}| \\ &\leq C \exp\left\{-\frac{z^2}{1 + |r'_{12}|}\right\} \leq C \exp\left\{-(1 - \epsilon)z^2\right\}, \end{aligned}$$

which, together with $Q_2(I_2; z) \le C\exp\{-z^2\}$, implies (S.43) for $d = 2$ with $\epsilon_2 = 1/2$ and some $C_2 > 1$. Now for $d \ge 3$, assume (S.43) holds for all dimensions less than d. There exists a matrix $(r'_{ij}) = \theta(r_{ij}) + (1-\theta)I_d$ for some $0 < \theta < 1$ such that

$$Q_d\left((r_{ij}); z\right) - Q_d\left((I_d; z\right) = \sum_{1\le h,l\le d} \frac{\partial Q_d}{\partial r_{hl}}((r'_{ij}); z, \ldots, z) r_{hl}. \tag{S.48}$$

By (S.40), $\mathbb{E}(X_k | X_h = X_l = z) \le 2\epsilon' z/(1-\epsilon')$ for $k \ne h, l$. Therefore, by writing the density in (S.39) as the product of the density of (X_h, X_l) and the conditional density of $\boldsymbol{X}_{-\{h,l\}}$ given $X_h = X_l = z$, where $\boldsymbol{X}_{-\{h,l\}}$ denotes the sub-vector $(X_1, \ldots, X_{h-1}, X_{h+1}, \ldots, X_{l-1}, X_{l+1}, \ldots, X_d)^\top$; we have

$$\left|\frac{\partial Q_d}{\partial r_{hl}}((r'_{ij}); z, \ldots, z)\right| \le \varphi_2((r'_{ij}); X_h = X_l = z) Q_{d-2}((r'_{ij|hl}); (1-3\epsilon)z), \tag{S.49}$$

where $(r'_{ij|hl})$ is the correlation matrix of the conditional distribution of $\boldsymbol{X}_{-\{h,l\}}$ given X_h and X_l. By (S.41) and (S.42), we know for $k, m \in [d] \setminus \{h, l\}$ and $k \ne m$,

$$\mathrm{Var}(X_k | X_h = X_l = z) \ge 1 - 3\epsilon^2 - 2\epsilon^3 \quad \text{and} \quad \mathrm{Cov}(X_k, X_m | X_h = X_l = z) \le \frac{\epsilon(1+\epsilon)}{1-\epsilon}.$$

Therefore, all the off-diagonal entries of $(r'_{ij|hl})$ are less than 2ϵ if we let $\epsilon < 1/5$. Applying the induction hypothesis, if $2\epsilon < \epsilon_{d-2}$, then

$$Q_{d-2}((r'_{ij|hl}); (1-3\epsilon)z) \le C_{d-2} \exp\left\{-\left(\frac{d-2}{2} - 2C_{d-2}\epsilon\right)(1-3\epsilon)^2 z^2\right\},$$

and equation (S.49) becomes

$$\begin{aligned}
&\left|\frac{\partial Q_d}{\partial r_{hl}}((r'_{ij}); z, \ldots, z)\right| \\
\le & CC_{d-2} \exp\left\{-(1-\epsilon)z^2\right\} \cdot \exp\left\{-\left(\frac{d-2}{2} - \left(2C_{d-2} + 3(d-2)\right)\epsilon\right) z^2\right\}.
\end{aligned}$$

Therefore, (S.43) holds for $\epsilon_d < \min\{1/5, \epsilon_{d-2}/2\}$ and some $C_d > 2C_{d-2} + 3(d-2) + 1$.

Using very similar arguments, inequality (S.45) can be proved by applying (S.43); and inequality (S.46) can be obtained by employing both (S.43) and (S.45). To prove inequality (S.44), which is a refinement of (S.43), it suffices to observe that, by (S.47), (S.48) and (S.49)

$$\begin{aligned} Q_d\left((r_{ij});z\right) &\le Q_d(I_d;z) + \sum_{1\le h,l\le d} C\,\epsilon\, \exp\{-(1-\epsilon)z^2\} Q_{d-2}((r'_{ij|hl});(1-3\epsilon)z) \\ &\le C_d \frac{1}{z^d} \exp\left\{\frac{dz^2}{2}\right\} + C_d\,\epsilon\, \exp\{-(1-\epsilon)z^2\} \sum_{1\le h,l\le d} Q_{d-2}((r'_{ij|hl});(1-3\epsilon)z); \end{aligned}$$

and apply the induction argument. □

Theorem 14. Let (X_n) be a stationary mean zero Gaussian process. Let $r_k = \mathrm{Cov}(X_0, X_k)$. Assume $r_0 = 1$, and $\lim_{n\to\infty} r_n(\log n) = 0$. Let $a_n = (2\log n)^{-1/2}$, $b_n = (2\log n)^{1/2} - (8\log n)^{-1/2}(\log\log n + \log 4\pi)$, and $z_n = a_n z + b_n$ for $z \in \mathbb{R}$. Define the event $A_i = \{X_i \ge z_n\}$, and

$$Q_{n,d} = \sum_{1\le i_1<\ldots<i_d\le n} P(A_{i_1}\cap\cdots\cap A_{i_d}).$$

Then $\lim_{n\to\infty} Q_{n,d} = e^{-dz}/d\,!$ for all $d \ge 1$. Furthermore, the same result holds if we define $A_i = \{|X_i| \ge z_{2n}\}$.

Proof of Theorem 14. Note that $z_n^2 = 2\log n - \log\log n - \log(4\pi) + 2z + o(1)$. If (X_n) consists of iid random variables, by the equality in (S.47),

$$\begin{aligned} \lim_{n\to\infty} Q_{n,d} &= \lim_{n\to\infty} \binom{n}{d} Q_d(I_d, z_n) \\ &= \lim_{n\to\infty} \binom{n}{d} \frac{1}{(2\pi)^{d/2} z_n^d} \exp\left\{-\frac{dz_n^2}{2}\right\} = \frac{e^{-dz}}{d!}. \end{aligned}$$

When the X_n's are dependent, the result is still trivially true when $d = 1$. Now we deal with the $d \ge 2$ case. Let $\gamma_k = \sup_{j\ge k} |r_j|$, then $\gamma_1 < 1$ by stationarity, and

$\lim_{n\to\infty}\gamma_n \log n = 0$. Consider an ordered subset

$$J = \{t, t+l_1, t+l_1+l_2, \ldots, t+l_1+\cdots+l_{d-1}\} \subset [n],$$

where $l_1, \ldots, l_{d-1} \geq 1$. We define an equivalence relation $\sim$ on J by saying $k \sim j$ if there exists $k_1, \ldots, k_p \in J$ such that $k = k_1 < k_2 < \cdots < k_p = j$, and $k_h - k_{h-1} \leq L$ for $2 \leq h \leq p$. For any $L \geq 2$, denote by $s(J, L)$ the number of l_j which are less than or equal to L. To similify the notation, we sometimes use s instead of $s(J, L)$. J is divided into $d-s$ equivalence classes $\mathcal{B}_1, \ldots, \mathcal{B}_{d-s}$. Suppose $s \geq 1$, assume w.l.o.g. that $|\mathcal{B}_1| \geq 2$. Pick $k_0, k_1 \in \mathcal{B}_1$, and $k_p \in \mathcal{B}_p$ for $2 \leq p \leq d-s$, and set $K = \{k_0, k_1, k_2, \ldots, k_{d-s}\}$. Define $Q_J = P(\cap_{k\in J} A_k)$ and Q_K similarly, then $Q_J \leq Q_K$. By (S.46) of Lemma S.5, there exists a number $M > 1$ depending on d and the sequence (γ_k), such that when $L > M$,

$$\begin{aligned} Q_K &\leq C_{d-s} \exp\left\{-\left(\frac{(1-\gamma_1)^2 + d - s}{2} - C_{d-s}\gamma_L\right) z_n^2\right\} \\ &\leq C_{d-s} \exp\left\{-\left(\frac{d-s}{2} + \frac{(1-\gamma_1)^2}{3}\right) z_n^2\right\}. \end{aligned}$$

Note that $z_n^2 = 2\log n - \log\log n + O(1)$. Pick $L_n = \max\{\lfloor n^\alpha \rfloor, M\}$ for some $\alpha < 2(1-\gamma_1^2)/3d$. For any $1 \leq a \leq d-1$, since there are at most $L_n^a n^{d-a}$ ordered subset $J \subset [n]$ such that $s(J, L_n) = a$, we know the sum of Q_J over these J is dominated by

$$C_{d-a} \exp\left\{\log n \left((d-a) + \frac{2(d-1)(1-\gamma_1)^2}{3d} - (d-a) - \frac{2(1-\gamma_1)^2}{3}\right)\right\}$$

when n is large enough, which converges to zero. Therefore, it suffices to consider all the ordered subsets J such that $l_j > L_n$ for all $1 \leq j \leq d-1$.

Let $J = \{t_1, \ldots, t_d\} \subset [n]$ be an ordered subset such that $t_i - t_{i-1} > L_n$ for $2 \leq i \leq d$, and $\mathcal{J}(d, L_n)$ be the collection of all such subsets. Let (r_{ij}) be the d-

dimensional covariance matrix of $\boldsymbol{X}_J$. There exists a matrix $R_J = \theta(r_{ij})_{i,j\in J} + (1-\theta)I_d$ for some $0 < \theta < 1$ such that

$$Q_J - Q_d(I_d, z_n) = \sum_{h,l\in J, h<l} \frac{\partial Q_d}{\partial r_{hl}}[R_J; z_n] r_{hl}.$$

Let R_H, $H = J \setminus \{h, l\}$, be the correlation matrix of the conditional distribution of $\boldsymbol{X}_H$ given X_h and X_l. By (S.44) of Lemma S.5, for n large enough

$$\begin{aligned}
\frac{\partial Q_d}{\partial r_{hl}}[R_J; z_n] &\le C \exp\left\{-\frac{z_n^2}{1+\gamma_{l-h}}\right\} \cdot Q_{d-2}\left(R_K; (1-3\gamma_{L_n})z_n\right) \\
&\le C C_{d-2} f_{d-2}(\gamma_{L_n}, 1/z_n) \exp\left\{-\frac{z_n^2}{1+\gamma_{l-h}}\right\} \\
&\quad \times \exp\left\{-\left(\frac{d-2}{2} - 2C_{d-2}\gamma_{L_n}\right)(1-3\gamma_{L_n}))^2 z_n^2\right\} \\
&\le C_d f_{d-2}(\gamma_{L_n}, 1/z_n) \exp\left\{-\left(\frac{d}{2} - (2C_{d-2} + 3(d-2))\gamma_{L_n} - \gamma_{h-l}\right) z_n^2\right\} \\
&\le C_d f_{d-2}(\gamma_{L_n}, 1/z_n) \exp\left\{-\left(\frac{d}{2} - C_d\gamma_{L_n} - \gamma_{h-l}\right) z_n^2\right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{J\in\mathcal{J}(d,L_n)} |Q_J - Q_d(I_d; z_n)| \\
&\le C_d f_{d-2}(\gamma_{L_n}, 1/z_n) \sum_{J\in\mathcal{J}(d,L_n)} \sum_{1\le i<j\le d} \exp\left\{-\left(\frac{d}{2} - C_d\gamma_{L_n} - \gamma_{t_j-t_i}\right) z_n^2\right\}\gamma_{t_j-t_i} \\
&= C_d f_{d-2}(\gamma_{L_n}, 1/z_n) \sum_{1\le i<j\le d} \sum_{J\in\mathcal{J}(d,L_n)} \exp\left\{-\left(\frac{d}{2} - C_d\gamma_{L_n} - \gamma_{t_j-t_i}\right) z_n^2\right\}\gamma_{t_j-t_i}.
\end{aligned} \tag{S.50}$$

For each fixed pair $1 \le i < j \le d$, the inner sum in (S.50) is bounded by

$$C_d f_{d-2}(\gamma_{L_n}, 1/z_n) \sum_{l=L_n+1}^{n-1} (n-l)^{d-1} \exp\left\{-\left(\frac{d}{2} - C_d\gamma_{L_n} - \gamma_l\right) z_n^2\right\}\gamma_l$$

$$\le C_d f_{d-2}(\gamma_{L_n}, 1/z_n)(\log n)^{d/2} n^{-d} \sum_{l=L_n+1}^{n-1} (n-l)^{d-1} \exp\left\{(C_d\gamma_{L_n} + \gamma_l)\, 2\log n\right\}\gamma_l \tag{S.51}$$

$$\le C_d f_{d-2}(\gamma_{\lfloor n^\alpha\rfloor}, 1/z_n)\, \gamma_{\lfloor n^\alpha\rfloor} (\log n)^{d/2} \exp\left\{2\left(C_d + 1\right)\gamma_{\lfloor n^\alpha\rfloor} \log n\right\}. \tag{S.52}$$

Since $\lim_{n\to\infty} \gamma_n \log n = 0$, it also holds that $\lim_{n\to\infty} \gamma_{\lfloor n^\alpha \rfloor} \log n = 0$. Note that $\lim_{n\to\infty} (\log n)^{1/2}/z_n = 2^{-1/2}$, it follows that $\lim_{n\to\infty} f_{d-2}(\gamma_{\lfloor n^\alpha \rfloor}, 1/z_n)(\log n)^{d/2-1} = 2^{-d/2+1}$. Therefore, the term in (S.52) converges to zero, and the proof of the first statement is complete.

Finally, observe that in the preceding proof, the upper bounds on Q_J and $|Q_J - Q(I_d; z_n)|$ are expressed through the absolute values of the correlations, so we can obtain the same bounds for probabilities of the form $P(\cap_{1\le i\le d}\{(-1)^{f_i} X_{t_i} \ge z_n\})$ for any $(f_1, \ldots, f_d) \in \{0,1\}^d$. The second statement follows from this observation. □

Remark S.3. This theorem provides another proof of Theorem 3.1 in Berman (1964), which gives the asymptotic distribution of the maximum term of a stationary Gaussian process. They also showed that the theorem is true if the condition $\lim_{n\to\infty} r_n \log n = 0$ is replaced by $\sum_{n=1}^{\infty} r_n^2 < \infty$. Under the later condition, if we replace $\gamma_{t_j - t_j}$ by $|r_{t_j - t_i}|$ in (S.50), γ_l by $|r_l|$ in (S.51), then the term in (S.51) converges to zero, and hence our result remains true.

S5 Summability of Cumulants

For a k-dimensional random vector $(Y_1, \ldots, Y_k)$ such that $\|Y_i\|_k < \infty$ for $1 \le i \le k$, the k-th order joint cumulant is defined as

$$\mathrm{Cum}(Y_1, \ldots, Y_k) = \sum (-1)^{p-1}(p-1)! \prod_{j=1}^{p} \left(\mathbb{E} \prod_{i\in\nu_j} Y_i \right), \tag{S.53}$$

where the summation extends over all partitions $\{\nu_1, \dots, \nu_p\}$ of the set $\{1, 2, \dots, k\}$ into p non-empty blocks. For a stationary process $(X_i)_{i\in\mathbb{Z}}$, we abbreviate

$$\gamma(k_1, k_2, \dots, k_d) := \mathrm{Cum}(X_0, X_{k_1}, X_{k_2}, \dots, X_{k_d}),$$

Summability conditions of cumulants are often assumed in the spectral analysis of time series, see for example Brillinger (2001) and Rosenblatt (1985). Recently, such conditions were used by Anderson and Zeitouni (2008) in studying the spectral properties of banded sample covariance matrices. While such conditions are true for some Gaussian processes, functions of Gaussian processes (Rosenblatt, 1985), and linear processes with iid innovations (Anderson, 1971), they are not easy to verify in general. Wu and Shao (2004) showed that the summability of joint cumulants of order d holds under the condition that $\delta_d(k) = O(\rho^k)$ for some $0 < \rho < 1$. We present in Theorem S.6 a generalization of their result. To simplify the proof, we introduce the composition of an integer. A *composition* of a positive integer n is an ordered sequence of strictly positive integers $\{\upsilon_1, \upsilon_2, \dots, \upsilon_q\}$ such that $\upsilon_1 + \cdots + \upsilon_q = n$. Two sequences that differ in the order of their terms define different compositions. There are in total 2^{n-1} different compositions of the integer n. For example, we are giving in the following all of the eight compositions of the integer 4.

$$\{1,1,1,1\} \quad \{1,1,2\} \quad \{1,2,1\} \quad \{1,3\} \quad \{2,1,1\} \quad \{2,2\} \quad \{3,1\} \quad \{4\}.$$

Theorem S.6. *Assume $d \geq 2$, $X_i \in \mathcal{L}^{d+1}$ and $\mathbb{E}X_i = 0$. If*

$$\sum_{k=0}^{\infty} k^{d-1}\delta_{d+1}(k) < \infty, \tag{S.54}$$

then

$$\sum_{k_1,\dots,k_d\in\mathbb{Z}} |\gamma(k_1, k_2, \dots, k_d)| < \infty. \tag{S.55}$$

Proof of Theorem S.6. By symmetry of the cumulant in its arguments and stationarity of the process, it suffices to show

$$\sum_{0\le k_1\le k_2\le\cdots\le k_d} |\gamma(k_1,k_2,\ldots,k_d)| < \infty.$$

Set $X(k,j) := \mathcal{H}_j X_k$, we claim

$$\begin{aligned}
&\gamma(k_1,k_2,\ldots,k_d)\\
=&\sum \mathrm{Cum}\big[X_0, X(k_1,1),\ldots,X(k_{\upsilon_1-1},1), X_{k_{\upsilon_1}} - X(k_{\upsilon_1},1),\\
&X(k_{\upsilon_1+1},k_{\upsilon_1}+1),\ldots,X(k_{\upsilon_2-1},k_{\upsilon_1}+1), X_{k_{\upsilon_2}} - X(k_{\upsilon_2},k_{\upsilon_1}+1),\\
&\cdots,\\
&X(k_{\upsilon_q+1},k_{\upsilon_q}+1),\ldots,X(k_{d-1},k_{\upsilon_q}+1), X_{k_d} - X(k_d,k_{\upsilon_q}+1)\big]\,; \qquad \text{(S.56)}
\end{aligned}$$

where the sum is taken over all the 2^{d-1} increasing sequences $\{\upsilon_0,\upsilon_1,\ldots,\upsilon_q,\upsilon_{q+1}\}$ such that $\upsilon_0 = 0$, $\upsilon_{q+1} = d$ and $\{\upsilon_1,\upsilon_2-\upsilon_1,\ldots,\upsilon_q-\upsilon_{q-1},d-\upsilon_q\}$ is a composition of the integer d. We first consider the last summand which corresponds to the sequence $\{\upsilon_0 = 0, \upsilon_1 = d\}$,

$$\mathrm{Cum}\,[X_0, X(k_1,1),\ldots,X(k_{d-1},1), X_{k_d} - X(k_d,1)]$$

Observe that X_0 and $(X(k_1,1),\ldots,X(k_{d-1},1))$ are independent. By definition, only partitions for which X_0 and $X_{k_d} - X(k_d,1)$ are in the same block contribute to the sum in (S.53). Suppose $\{\nu_1,\ldots,\nu_p\}$ is a partition of the set $\{k_1,k_2,\ldots,k_{d-1}\}$, since

$$\begin{aligned}
\left|\mathbb{E}\left[X_0(X_{k_d} - X(k_d,1))\prod_{k\in\nu_1} X(k,1)\right]\right| &= \left|\sum_{j=-\infty}^{0}\mathbb{E}\left[\mathcal{P}_j X_0 \mathcal{P}_j X_{k_d}\prod_{k\in\nu_1} X(k,1)\right]\right|\\
&\le \sum_{j=-\infty}^{0}\delta_{d+1}(-j)\delta_{d+1}(k_d-j)\kappa_{d+1}^{|\nu_1|},
\end{aligned}$$

it follows that

$$\begin{aligned}&\left|\mathbb{E}\left[X_0(X_{k_d}-X(k_d,1))\prod_{k\in\nu_1}X(k,1)\right]\cdot\prod_{j=2}^{p}\left(\mathbb{E}\prod_{k\in\nu_j}X(k,1)\right)\right|\\ \leq&\sum_{j=0}^{\infty}\delta_{d+1}(j)\delta_{d+1}(k_d+j)\kappa_{d+1}^{d-1}\end{aligned}$$

and therefore

$$\begin{aligned}&\sum_{0\leq k_1\leq k_2\leq\cdots\leq k_d}\left|\mathrm{Cum}\left[X_0,X(k_1,1),\ldots,X(k_{d-1},1),X_{k_d}-X(k_d,1)\right]\right|\\ \leq&C_d\sum_{0\leq k_1\leq k_2\leq\cdots\leq k_d}\sum_{j=0}^{\infty}\delta_{d+1}(j)\delta_{d+1}(k_d+j)\\ \leq&C_d\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+d-1}{d-1}\delta_{d+1}(j)\delta_{d+1}(k+j)<\infty,\end{aligned}$$

provided that $\sum_{k=0}^{\infty}k^{d-1}\delta_{d+1}(k)<\infty$.

The other terms in (S.56) are easier to deal with. For example, for the term corresponding to the sequence $\{\upsilon_0=0,\upsilon_1=1,\upsilon_2=d\}$, we have

$$\begin{aligned}&\left|\mathrm{Cum}\left[X_0,X_{k_1}-X(k_1,1),X(k_2,k_1+1),\ldots,X(k_{d-1},k_1+1),X_{k_d}-X(k_d,k_1+1)\right]\right|\\ &\leq C_d\kappa_{d+1}^{d-1}\Psi_{d+1}(k_1)\Psi_{d+1}(k_d-k_1).\end{aligned}$$

Since $\sum_{k=0}^{\infty}k^{d-1}\delta_{d+1}(k)<\infty$ implies $\sum_{k=0}^{\infty}k^{d-2}\Psi_{d+1}(k)\leq\infty$, it follows that

$$\begin{aligned}&\sum_{0\leq k_1\leq k_2\leq\cdots\leq k_d}\left|\mathrm{Cum}\left[X_0,X_{k_1}-X(k_1,1),X(k_2,k_1+1),\ldots,\right.\right.\\ &\qquad\qquad\left.\left.X(k_{d-1},k_1+1),X_{k_d}-X(k_d,k_1+1)\right]\right|\\ \leq&C_d\kappa_{d+1}^{d-1}\sum_{k=0}^{\infty}\Psi_{d+1}(k)\sum_{k=0}^{\infty}\binom{k+d-2}{d-2}\Psi_{d+1}(k)\leq\infty.\end{aligned}$$

We have shown that every cumulant in (S.56) is absolutely summable over $0\leq k_1\leq\cdots\leq k_d$, and it remains to show the claim (S.56). We shall derive the case $d=3$, (S.56) for other values of d are obtained using the same idea. By multilinearity of cumulants, we have

$$\begin{aligned}\gamma(k_1,k_2,k_3) &= \mathrm{Cum}(X_0, X_{k_1}, X_{k_2}, X_{k_3}) \\ &= \mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X_{k_2}, X_{k_3}\right] \\ &\quad + \mathrm{Cum}\left[X_0, X(k_1,1), X_{k_2} - X(k_2,1), X_{k_3}\right] \\ &\quad + \mathrm{Cum}\left[X_0, X(k_1,1), X(k_2,1), X_{k_3} - X(k_3,1)\right] \\ &\quad + \mathrm{Cum}\left[X_0, X(k_1,1), X(k_2,1), X(k_3,1)\right].\end{aligned}$$

Since X_0 and $(X(k_1,1), X(k_2,1), X(k_3,1))$ are independent, the last cumulant is 0. Apply the same trick for the first two cumulants, we have

$$\begin{aligned}&\mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X_{k_2}, X_{k_3}\right] \\ &= \mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X_{k_2} - X(k_2, k_1+1), X_{k_3}\right] \\ &+ \mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X(k_2, k_1+1), X_{k_3} - X(k_3, k_1+1)\right] \\ &+ \mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X(k_2, k_1+1), X(k_3, k_1+1)\right] \\ &= \mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X_{k_2} - X(k_2, k_1+1), X_{k_3} - X(k_3, k_2+1)\right] \\ &+ \mathrm{Cum}\left[X_0, X_{k_1} - X(k_1,1), X(k_2, k_1+1), X_{k_3} - X(k_3, k_1+1)\right]\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Cum}\left[X_0, X(k_1,1), X_{k_2} - X(k_2,1), X_{k_3}\right] \\ &= \mathrm{Cum}\left[X_0, X(k_1,1), X_{k_2} - X(k_2,1), X_{k_3} - X(k_3, k_2+1)\right].\end{aligned}$$

Then the proof is complete. □

Remark S.4. When $d = 1$, (S.54) reduces to the *short-range dependence* or *short-memory* condition $\Theta_2 = \sum_{k=0}^{\infty} \delta_2(k) < \infty$. If $\Theta_2 = \infty$, then the process (X_i) may be long-memory in that the covariances are not summable. When $d \geq 2$, we conjecture that (S.54) can be weakened to $\Theta_{d+1} < \infty$. It holds for linear processes. Let $X_k = \sum_{i=0}^{\infty} a_i \epsilon_{k-i}$. Assume $\epsilon_k \in \mathcal{L}^{d+1}$ and $\sum_{k=0}^{\infty} |a_k| < \infty$, then $\delta_{d+1}(k) = |a_k| \|\epsilon_0\|_{d+1}$. Let $\mathrm{Cum}_{d+1}(\epsilon_0)$ be the $(d+1)$-th cumulant of ϵ_0. Set $k_0 = 0$, by multilinearity of cumulants, we have

$$\begin{aligned} \gamma(k_1, \ldots, k_d) &= \sum_{t_0, t_1, \ldots, t_d \geq 0} \left(\prod_{j=0}^{d} a_{t_j} \right) \mathrm{Cum}(\epsilon_{-t_0}, \epsilon_{k_1 - t_1}, \ldots, \epsilon_{k_d - t_d}) \\ &= \sum_{t=0}^{\infty} \prod_{j=0}^{d} a_{k_j + t} \, \mathrm{Cum}_{d+1}(\epsilon_0). \end{aligned}$$

Therefore, the condition $\Theta_{d+1} < \infty$ suffices for (S.55). For a class of functionals of Gaussian processes, Rosenblatt (1985) showed that (S.55) holds if $\sum_{k=0}^{\infty} |\gamma_k| < \infty$, which in turn is implied by $\Theta_{d+1} < \infty$ under our setting. It is unclear whether in general the weaker condition $\Theta_{d+1} < \infty$ implies (S.55).

S6 Auxiliary Results

In this section we collect several auxiliary results. Suppose that $\boldsymbol{X}$ is a d-dimensional random vector, and $\boldsymbol{X} \sim \mathcal{N}(0, \Sigma)$. If $\Sigma = I_d$, then by (S.47), it is easily seen that the ratio of $P\left(z_n - c_n \leq |\boldsymbol{X}|_\bullet \leq z_n\right)$ over $P\left(|\boldsymbol{X}|_\bullet \geq z_n\right)$ tends to zero provided that $c_n \to 0$, $z_n \to \infty$ and $c_n z_n \to 0$. It is a similar situation when Σ is not an identity matrix, as shown in Lemma S.7, which will be used in the proof of Lemma 13.

Lemma S.7. *Let $\boldsymbol{X} \sim \mathcal{N}(0, \Sigma)$ be a d-dimensional normal random vector. Assume Σ is nonsingular. Let λ_0^2 and λ_1^2 be the smallest and largest eigenvalue of Σ respectively. Then for $0 < c < \delta < 1/2$ such that $A := (2\pi\lambda_1^2)^{(d-1)/2} \lambda_0^2 c^2 \delta^{-2} + d\delta \exp\{(\sqrt{6} d\lambda_1 + \lambda_0)/\lambda_0^3\} < 1$,* *then for any* $z \in [1, \delta/c]$,

$$P\left(z - c \le \|\boldsymbol{X}\|_\bullet \le z\right) \le (1-A)^{-1} A\, P\left(\|\boldsymbol{X}\|_\bullet \ge z\right). \tag{S.57}$$

Proof of Lemma S.7. Let $C_d = (6d)^{1/2}\lambda_1/\lambda_0$. Since λ_0^2 is the smallest eigenvalue of Σ,

$$\begin{aligned} P(\|\boldsymbol{X}\|_\bullet \ge z - c) &\ge (2\pi \det(\Sigma))^{-d/2} \exp\left\{-\frac{d(z+1)^2}{2\lambda_0^2}\right\} \\ &\ge (2\pi\lambda_1^2)^{-d/2} \exp\left\{-\frac{4d\delta^2}{2\lambda_0^2 c^2}\right\}. \end{aligned}$$

Since $P(\|\boldsymbol{X}\|_\infty \ge C_d\delta/c) \le d(2\pi\lambda_1^2)^{-1/2} \exp\{6d\delta^2/(2\lambda_0^2c^2)\}$, we have

$$P(\|\boldsymbol{X}\|_\infty \ge C_d\delta/c) \le (2\pi\lambda_1^2)^{(d-1)/2}\lambda_0^2 c^2\delta^{-2}\, P(\|\boldsymbol{X}\|_\bullet \ge z-c). \tag{S.58}$$

For $0 \le k \le \lfloor 1/\delta \rfloor$, define the orthotopes $R_k = [z+(k-1)c, z+kc] \times [z-c, C_d\delta/c]^{d-1}$. For two points $\boldsymbol{x} = (x_1, \ldots, x_d) \in R_0$, $\boldsymbol{x}_k = (x_1 + kc, x_2, \ldots, x_d) \in R_k$, we have $\boldsymbol{x}_k^\top \Sigma^{-1}\boldsymbol{x}_k - \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x} \le (2\sqrt{d}C_d + 1)/\lambda_0^2$, and hence $P(\boldsymbol{X} \in R_k) \ge \exp\{-(\sqrt{d}C_d+1)/\lambda_0^2\} P(\boldsymbol{X} \in R_0)$ for any $1 \le k \le \lfloor 1/\delta \rfloor$. Since the same inequality holds for every coordinate, we have

$$P\left(z - c \le \|\boldsymbol{X}\|_\bullet \le z,\, \|\boldsymbol{X}\|_\infty \le C_d\delta/c\right) \le d\delta \exp\{(\sqrt{d}C_d + 1)/\lambda_0^2\}\, P\left(\|\boldsymbol{X}\|_\bullet \ge z - c\right) \tag{S.59}$$

Combine (S.58) and (S.59), we know $P\left(z - c \le \|\boldsymbol{X}\|_\bullet \le z\right) \le A \cdot P\left(\|\boldsymbol{X}\|_\bullet \ge z - c\right)$. So (S.57) follows. □

Lemma S.7 requires the eigenvalues of Σ to be bounded both from above and away from zero. In our application, Σ is taken as the covariance matrix of $(G_{k_1}, G_{k_2}, \ldots, G_{k_d})^\top$, where (G_k) is defined in (6). Furthermore, we need such bounds be uniform over all choices of $k_1 < k_2 < \cdots < k_d$. Let $f(\omega) = (2\pi)^{-1}\sum_{h\in\mathbb{Z}} \sigma_h \cos(h\omega)$ be the spectral density of (G_k). A sufficient condition would be that there exists $0 < m < M$ such that

$$m \le f(\omega) \le M, \quad \text{for } \omega \in [0, 2\pi], \tag{S.60}$$

because the eigenvalues of the autocovariance matrix are bounded from above and below by the maximum and minimum values that f takes respectively. For the proof see Section 5.2 of Grenander and Szegö (1958). Clearly the upper bound in (S.60) is satisfied in our situation, because $\sum_{h\in\mathbb{Z}}|\sigma_h| < \infty$. However, the existence of lower bound in (S.60) rules out some classical times series models. For example, if (G_k) is the moving average of the form $G_k = (\eta_k + \eta_{k-1})/\sqrt{2}$, then $f(\omega) = (1+\cos(\omega))/2\pi$, and $f(\pi) = 0$. Nevertheless, although the minimum eigenvalue of the autocovariance matrix converges to $\inf_{\omega\in[0,2\pi]} f(\omega)$ as the dimension of the matrix goes to infinity, there does exist a positive lower bound for the smallest eigenvalues of all the principal sub-matrices with a fixed dimension, as stated in Lemma S.8.

Lemma S.8. *If $0 < \sum_{h\in\mathbb{Z}} \sigma_h^2 < \infty$, then for each $d \geq 1$, there exists a constant $C_d > 0$ such that*

$$\inf_{k_1<k_2<\cdots<k_d} \lambda_{\min}\left\{\operatorname{Cov}\left[(G_{k_1}, G_{k_2}, \ldots, G_{k_d})^\top\right]\right\} \geq C_d.$$

Proof of Lemma S.8. We use induction. It is clear that we can choose (C_d) to be a non-increasing sequence. Without loss of generality, let us assume $k_1 = 1$. The statement is trivially true when $d = 1$. Suppose it is true for all dimensions up to d, we now consider the dimension $(d+1)$ case. There exist an integer N_d such that $\sum_{h=N_d} \sigma_h^2 < 2C_d^2/(d+1)$. If all the differences $k_{i+1} - k_i \leq N_d$ for $1 \leq i \leq d-1$, there are N_d^{d-1} possible choices of $k_1 = 1 < k_2 < \cdots < k_d$. Since the process (G_k) is non-deterministic, for all these choices, the corresponding covariance matrices are non-singular. Pick $C_d' > 0$ to be the smallest eigenvalue of all these matrices. If there is one difference $k_{l+1} - k_l > N_d$, set $\Sigma_1 = \operatorname{Cov}[(G_{k_i})_{1\leq i\leq l}]$ and $\Sigma_2 = \operatorname{Cov}[(G_{k_i})_{l<i\leq d}]$, then $\lambda_{\min}(\Sigma_1) \geq C_d$ and $\lambda_{\min}(\Sigma_2) \geq$

C_d. It follows that for any real numbers $c_1, c_2, \ldots, c_d$ such that $\sum_{i=1}^d c_i^2 = 1$,

$$\begin{aligned}\sum_{1\le i,j\le d} c_i c_j \operatorname{Cov}(G_{k_i}, G_{k_j}) &= (c_1, \ldots, c_i)^\top \Sigma_J (c_1, \ldots, c_i)\\ &\quad +(c_{i+1}, \ldots, c_d)^\top \Sigma_J (c_{i+1}, \ldots, c_d)\\ &\quad +2 \sum_{i\le l, j>l} c_i c_j \sigma_{k_j - k_i}\\ &\ge C_d - 2\left(\sum_{i\le l,j>l} \sigma_{k_j-k_i}^2\right)^{1/2} \left(\sum_{i\le l, j>l} c_i^2 c_j^2\right)^{1/2}\\ &\ge C_d - \frac{1}{2}\left(\frac{d+1}{2}\cdot \sum_{h=N_d} \sigma_h^2\right)^{1/2} \ge \frac{C_d}{2}.\end{aligned}$$

Setting $C_{d+1} = \min\{C_d/2, C_d'\}$, the proof is complete. □

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