960:379– Basic Probability and Statistics– Fall, 2010

Homework 4 Solutions,

1. Question 4.1.4.

a. This question asks about a random variable $X \sim U(0, 5/2)$ *. From part b we can assume these quantities are measured in meters. From the formula in the book,* $\mathrm{E}[X] = (5/2 - 0)/2 = 5/4m$ *, and* $Var[X] = (5/2 - 0)^2/12 = 25/48m^2$ *.*

b. This question involves both the uniform and the binomial distribution. Note that the probability that any one piece of wood has length exceeding 1 meter. Assume these quantities are measured in meters. $P(X > 1) = 3/5$ *. Let* Y *be the count of boards exceeding 1 meter. Then* $Y \sim Bin(25, 3/5)$ *, and*

$$
\begin{aligned}
&P(Y \geq 20)\\
&= \binom{25}{20}\left(\frac{3}{5}\right)^{20}\left(\frac{2}{5}\right)^{5} + \binom{25}{21}\left(\frac{3}{5}\right)^{21}\left(\frac{2}{5}\right)^{4}\\
&+ \binom{25}{22}\left(\frac{3}{5}\right)^{22}\left(\frac{2}{5}\right)^{3} + \binom{25}{23}\left(\frac{3}{5}\right)^{23}\left(\frac{2}{5}\right)^{2}\\
&+ \binom{25}{24}\left(\frac{3}{5}\right)^{24}\left(\frac{2}{5}\right)^{1} + \binom{25}{25}\left(\frac{3}{5}\right)^{25}\left(\frac{2}{5}\right)^{0}\\
&= \frac{25 \times 24 \times 23 \times 22 \times 21}{120}\left(\frac{3}{5}\right)^{20}\left(\frac{2}{5}\right)^{5} + \frac{25 \times 24 \times 23 \times 22}{24}\left(\frac{3}{5}\right)^{21}\left(\frac{2}{5}\right)^{4}\\
&+ \frac{25 \times 24 \times 23}{6}\left(\frac{3}{5}\right)^{22}\left(\frac{2}{5}\right)^{3}\\
&+ \frac{25 \times 24}{2}\left(\frac{3}{5}\right)^{23}\left(\frac{2}{5}\right)^{2} + 25\left(\frac{3}{5}\right)^{24}\left(\frac{2}{5}\right)^{1} + \left(\frac{3}{5}\right)^{25}\left(\frac{2}{5}\right)^{0}\\
&= 3^{20}5^{-25}(5 \times 23 \times 22 \times 21 \times 2^5 + 25 \times 23 \times 22 \times 3^1 \times 2^4\\
&+ 25 \times 4 \times 23 \times 3^2 \times 4^3 + 25 \times 12 \times 3^3 \times 4^2 + 25 \times 3^4 \times 4 + 3^5)\\
&= 0.029.
\end{aligned}
$$

This might more easily be calculated by `1-pbinom(19,25,.6)`.

2. Question 4.2.2.

a. If $X \sim Expon(.1)$ *, then* $\mathrm{E}[X] = 1/.1 = 10$ *.*
b. If $P(X \geq 10) = 1 - P(X < 10) = 1 - (1 - \exp(-.1 * 10)) = \exp(-1) = 0.368$ *.*
c. If $P(X \leq 5) = 1 - \exp(-.1 * 5) = 0.393$ *.*
d. By the memoryless property of the exponential distribution, the additional waiting time is $Expon(.1)$ *, and so* $P(X \geq 15 | X > 5) = P(X \geq 10) = 0.368$ *.*
e. If $X \sim U(0, 20)$ *, then* $\mathrm{E}[X] = (0 + 20)/2 = 10$ *, as before. Also, for any* $x \in [5, 20]$ *, then* $P(X \leq x | X \geq 5) = P(X \in [5, x]) / P(X > 5) = \frac{(x-5)/20}{(20-5)/20} = (x - 5)/(20 - 5)$ *, and so the conditional distribution is* $U(5, 20)$ *.*

3. Question 4.3.4.

a. Since $X \sim \Gamma(5, 0.9)$ *, then* $\mathrm{E}[X] = 5/0.9 = 5.55$ *.*
b. Since $Var[X] = 5/0.9^2 = 6.17$ *, the standard deviation is* $\sqrt{6.17} = 2.48$ *.*
c. You'll need to use R or something like it for this. `qgamma(.25,5,.9)= 3.7429`.

d. You'll need to use R or something like it for this. `1-pgamma(6,5,.9)= 0.37331`.

4. Question 4.4.4.

a. Use the relationship with the exponential distribution: $(.5X)^3$ *Expon*(1) *, and so the median of* $(.5X)^3$ *is* $\ln(2)$ *, so the median of* X *is* $\ln(2)^{1/3}/.5 = 1.7700$ *.*

b. Again ue the relationship with the exponential distribution. The .99 quantile of the exponential distribution satisfies $1 - \exp(-q) = .99$ *, or* $q = -\log(.01) = \log(100) = 4.605$ *. So the* $.99$ *quantile of* X *is* $\log(100)^{1/3}/.5 = 3.327$ *.*

c. $\mathrm{E}[X] = (0.5)^{-1}\Gamma(1 + 1/3)$ *.*

d. The probability that one circuit will be working after 3 hours is $\exp(-(0.5 \times 3)^3) = 0.034$ *. The probability that one circuit will fail before 3 hours is* 0.966 *. The probability that all four will fail before 3 hours is* $(.966)^4 = .871$ *. The probability that at least one circuit will still be working after 3 hours is* $1 - .871 = .129$ *.*

5. Question 4.5.6.

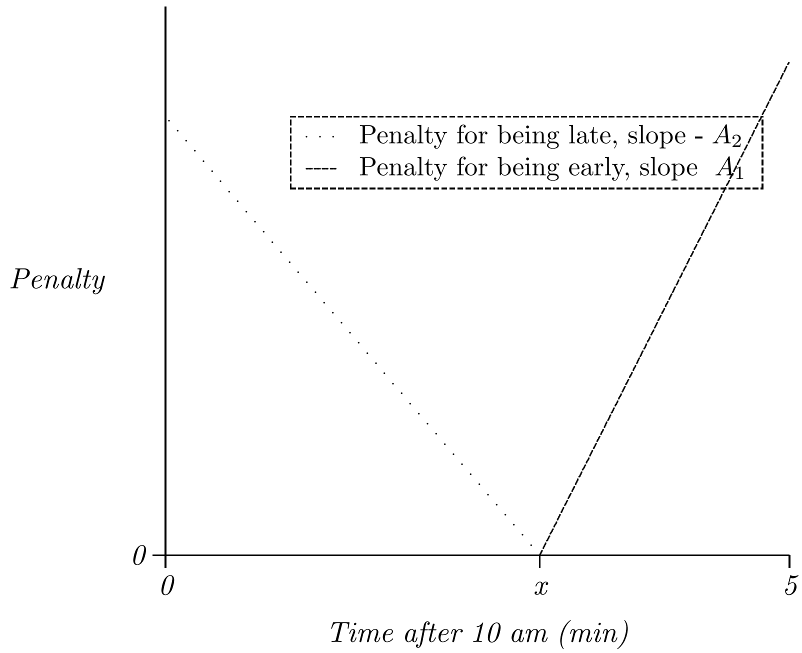
a. If $X \sim Beta(8.2, 11.7)$ *then* $\mathrm{E}[X] = 8.2/(8.2 + 11.7) = 0.412$ *.*

b. If $X \sim Beta(8.2, 11.7)$ *then* $Var[X] = (8.2 \times 11.7)/((8.2 + 11.7)^2(8.2 + 11.7 + 1)) = 0.0116$ *, and the standard deviation is* $\sqrt{0.0116} = 0.108$ *.*

c. If $X \sim Beta(8.2, 11.7)$ *then use R to get the median; do* `qbeta(.5,8.2,11.7)=0.409`.

6. Question 4.7.8.

Let S *be the class starting time, in minutes after 10:00 AM. If a student adopts the policy of walking in at* x *minutes after 10, then the penalty is* $Y = g(S)$ *, for* $g(s) = \begin{cases} A_1(s-x) & \text{if } s > x \\ A_2(x-s) & \text{if } s < x \end{cases}$ *. The expectation of* Y *, as a function of* x *, is then* $\mathrm{E}[Y] = \int_0^5 g(s)\,ds/5 = \int_x^5 A_1(s-x)\,ds/5 + \int_0^x A_2(x-s)\,dx/5 = A_1(s^2/2 - xs)/5\big|_x^5 + A_2(xs - s^2/2)/5\big|_0^x = (12.5 - 5x + x^2/2)A_1/5 + A_2(x^2/2)/5$ *. Call this* $h(x)$ *. Then* $h(x)$ *is minimized where* $h'(x) = -A_1 + xA_1/5 + A_2x/5 = 0$ *, or* $x = 5A_1/(A_1 + A_2)$ *. We confirm this is a minimum by noting that* $h'(x) = (A_1 + A_2)/5 > 0$ *.*



Penalty for being late, slope - A_2
Penalty for being early, slope A_1
Penalty
0
0
x
5
Time after 10 am (min)