$$\begin{aligned}\mathrm{E}[X^2] &= \int_0^\infty x^2 x^{k-1}\lambda^k \exp(\lambda x)/\Gamma(k)\,dx \\ &= \lambda^{-2}\int_0^\infty x^{k+1}\lambda^{k+2}\exp(\lambda x)/\Gamma(k)\,dx \\ &= \lambda^{-2}\Gamma(k+2)/\Gamma(k) = (k+1)k\lambda^{-2}.\end{aligned}$$

Var $[X] = (k+1)k\lambda^{-2} - k^2\lambda^{-2} = k\lambda^{-2}$.

: 4.4-4.5

C. Weibull Distribution: Another variant on the Exponential frequently used to model failure times.

1. Raise X to a power: X Weib(a, λ) if and only if $Y = (\lambda X)^a \sim$ Expon(1), for $a > 0$ and $\lambda > 0$.
2. Then $F_X(x) = P(X \le x) = P((\lambda X)^a \le (\lambda x)^a) = 1 - \exp(-(\lambda x)^a)$.
3. Then $f_X(x) = \frac{d}{dx}(1 - \exp(-(\lambda x)^a)) = \exp(-(\lambda x)^a)a(\lambda x)^{a-1}\lambda$.
4. Then

$$\begin{aligned}\mathrm{E}[X] &= \int_0^\infty x\exp(-(\lambda x)^a)a(\lambda x)^{a-1}\lambda\,dx \\ &= \int_0^\infty a\exp(-(\lambda x)^a)(\lambda x)^a\,dx \\ &= \int_0^\infty \exp(-z)az(1/a)z^{1/a-1}\lambda^{-1}\,dx \\ &= \lambda^{-1}\int_0^\infty \exp(-z)z^{1/a}\,dx \\ &= \lambda^{-1}\Gamma(1+1/a).\end{aligned}$$

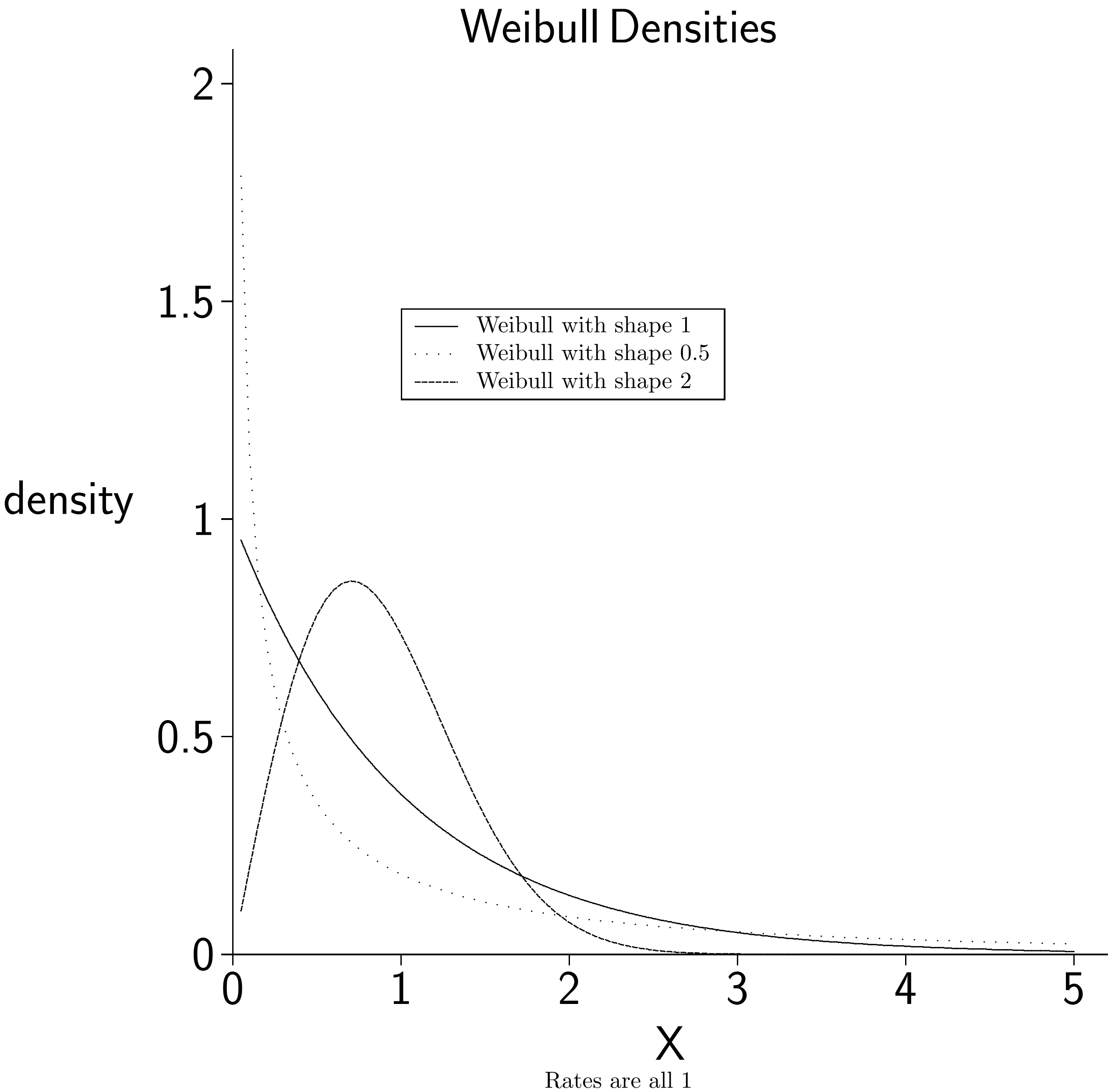
5. Similarly $\mathrm{E}[X^2] = \lambda^{-2}\Gamma(1+2/a)$.
6. Hence Var $[X] = \lambda^{-2}(\Gamma(1+2/a) - \Gamma(1+1/a)^2)$. with $z = (\lambda x)^a$, and hence $x = z^{1/a}/\lambda$.
7. Since the map from X to Y is strictly increasing, if y satisfies $P(Y \le y) = p$, then $P\left(X \le y^{1/a}/\lambda\right) = p$.
   a. Also note that the p quantile for Y satisfies $\exp(-y) = 1-p$, or $y = -\log(1-p)$.
   b. Hence the p quantile of X is $(-\log(1-p))^{1/a}/\lambda$.

D. Beta distribution for $a > 0$, $b > 0$:

1. $f_X(x) \propto x^{a-1}(1-x)^{b-1}$ for $x \in (0,1)$.
2. Let $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$
3. $f_X(x) = x^{a-1}(1-x)^{b-1}/B(a,b)$ for $x \in (0,1)$.
4. $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$
   a. To see this, check $B(a,b)\Gamma(a+b) = \Gamma(a)\Gamma(b)$.
   b. Write right hand side as

$$\int_0^\infty \int_0^\infty x^{a-1}\exp(-x)y^{b-1}\exp(-y)\,dx\,dy$$

   c. Reparameterize to $z = x+y$ and $w = x/(x+y)$.


Weibull Densities
density
2
1.5
1
0.5
0
0
1
2
3
4
5
X
Weibull with shape 1
Weibull with shape 0.5
Weibull with shape 2
Rates are all 1

5. Expectation:

$$\begin{aligned}
\mathrm{E}[X] &= \textstyle\int_0^1 x x^{a-1}(1-x)^{b-1}\,dx/B(a,b) \\
&= \textstyle\int_0^1 x^{(a+1)-1}(1-x)^{b-1}\,dx/B(a,b) \\
&= B(a+1,b)/B(a,b) \\
&= \frac{\Gamma(a+1)\Gamma(b)/\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)/\Gamma(a+b)} \\
&= a/(a+b)
\end{aligned}$$

6. Variance:

$$\begin{aligned}
\mathrm{E}[X^2] &= \textstyle\int_0^1 x^2 x^{a-1}(1-x)^{b-1}\,dx/B(a,b) \\
&= \textstyle\int_0^1 x^{(a+2)-1}(1-x)^{b-1}\,dx/B(a,b) \\
&= B(a+2,b)/B(a,b) \\
&= \frac{\Gamma(a+2)\Gamma(b)/\Gamma(a+b+2)}{\Gamma(a)\Gamma(b)/\Gamma(a+b)} \\
&= (a+1)a/((a+b+1)(a+b))
\end{aligned}$$

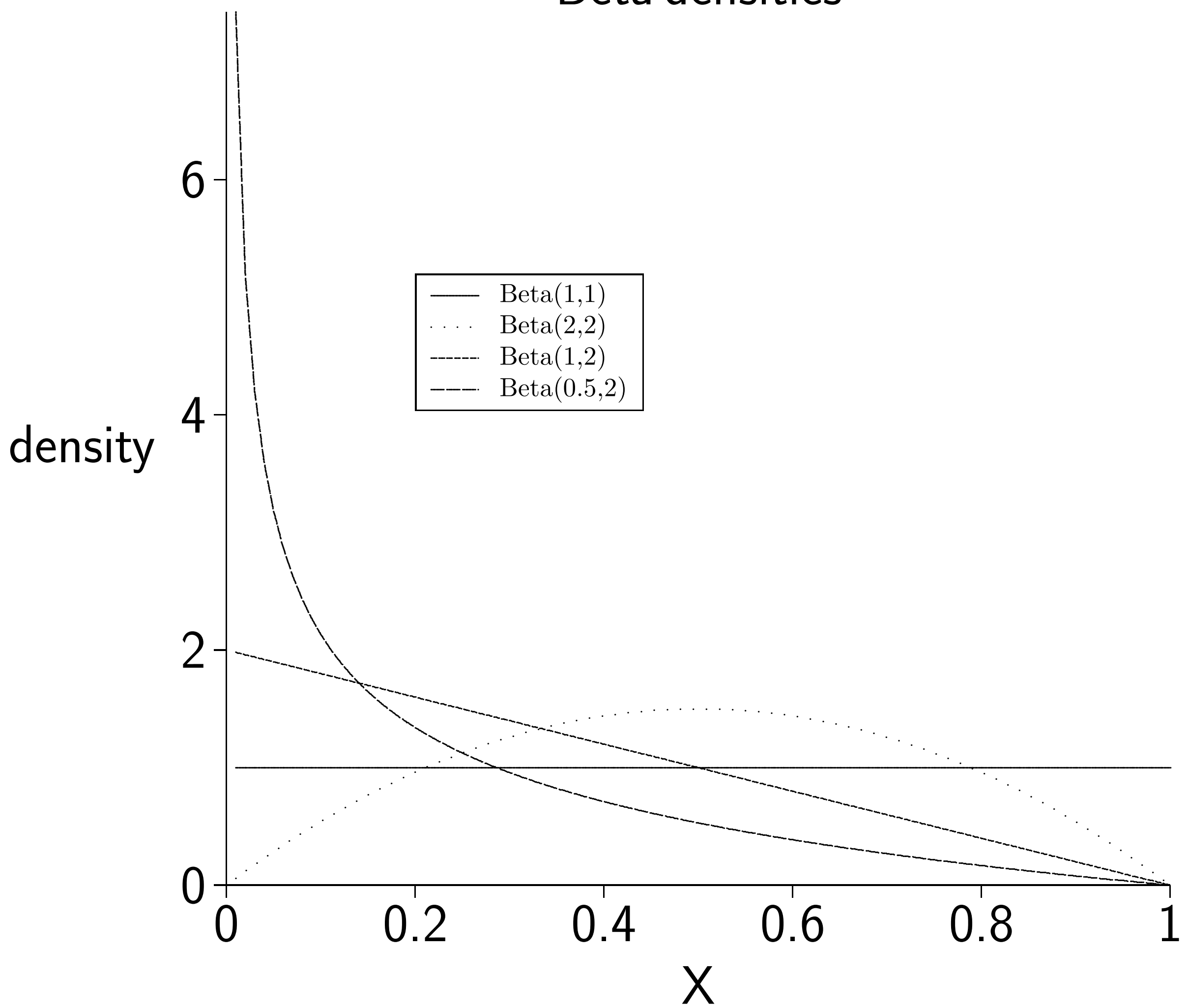
$$\begin{aligned}
\text{Var}\,[X] &= (a+1)a/((a+b+1)(a+b)) - (a/(a+b))^2 \\
&= \frac{(a+1)a(a+b) - a^2(a+b+1)}{(a+b+1)(a+b)^2} \\
&= \frac{ab}{(a+b+1)(a+b)^2}
\end{aligned}$$

7. Special case: $a = b = 1$ gives uniform.

: 5.1

E. The Gaussian, or Normal, Distribution:

Beta densities



1. pdf $A^{-1}\sigma^{-1}\exp(-(x-\mu)^2/(2\sigma^2))$
   a. To calculate A, note that $A = \int_{-\infty}^{\infty} \sigma^{-1}\exp(-(x-\mu)^2/(2\sigma^2))\,dx = \int_{-\infty}^{\infty}\exp(-z^2/2))\,dz$, which is free of μ and σ.