b. $A = 2^{1/2} \int_0^\infty \exp(-w) w^{-1/2}\, dw$ for $w = z^2/2$, $z = \sqrt{2w}$, $dz = 2^{-1/2} w^{-1/2}\, dw$.

c.

$$
\begin{aligned}
A^2 &= 2 \int_0^\infty \int_0^\infty \frac{\exp(-w-v)}{(vw)^{1/2}}\, dv\, dw. \\
&= 2 \int_0^\infty \int_0^u \frac{\exp(-u)}{(v(u-v))^{1/2}}\, dv\, du \text{ using } u = w+v. \\
&= 2 \int_0^\infty \int_0^1 \frac{\exp(-u)}{u(s(1-s))^{1/2}} u\, ds\, du \text{ using } s = v/u. \\
&= 2 \int_0^\infty \exp(-u)\, du \int_0^1 (s(1-s))^{-1/2}\, ds \\
&= 2 \times 1 \times \int_0^{\pi/2} \frac{2\sin(t)\cos(t)}{\sin(t)\cos(t)}\, dt \text{ using } s = \sin^2(t) \\
&= 4 \int_0^{\pi/2} dt = 2\pi \\
A &= \sqrt{2\pi}
\end{aligned}
$$

2. Clearly if $X \sim \mathrm{N}(\mu, \sigma^2)$ then the median of X is μ, by symmetry.

3. $\mathrm{E}[X] = \mu$, again by symmetry, if expectation exists

   a. That is, if $\int_{-\infty}^\infty |x| \sigma^{-1} (2\pi)^{-1/2} \exp(-(x-\mu)^2/(2\sigma^2))\, dx < \infty$

   b. Integral is finite, by comparing with integral with $-|\cdot|$ replacing $-(\cdot)^2$ in exponent.

4.

$$\text{Var}[X] = \int_{-\infty}^{\infty} \frac{(x-\mu)^2 \exp(-(x-\mu)^2/(2\sigma^2))}{\sigma(2\pi)^{1/2}}\, dx$$
$$= \sigma^2 \int_{-\infty}^{\infty} z^2 (2\pi)^{-1/2} \exp(-z^2/2))\, dz$$

and use integration by parts, with $u = z$ and $v = \exp(-z^2/2))$ to show that the integral is 1.

5. Note that many of these explorations began by changing variables to the case with $\mu = 0$, and $\sigma = 1$.

a. This case is known as *standard normal*.

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b. If $X \sim \mathsf{N}(\mu, \sigma^2)$ and $Y = aX + b$ for $a \neq 0$ then $Y \sim \mathsf{N}(a\mu + b, a^2\sigma^2)$

i. Here $Y = g(X)$ for $g(x) = ax+b$, and $g^{-1}(y) = (y-b)/a$.

ii. Use rule $f_Y(y) = f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = \frac{\exp(-((y-b)/a-\mu)^2/(2\sigma^2))}{\sqrt{2\pi}\sigma}a^{-1} = \frac{\exp(-(y-(b+a\mu))^2/(2a^2\sigma^2))}{\sqrt{2\pi}\sigma a}$

c. If $X \sim \mathsf{N}(\mu, \sigma^2)$ then $Y = (X-\mu)/\sigma \sim \mathsf{N}(0, 1)$.

d. Then $F_X(x) = P(X \leq x) = P(Z \leq (x-\mu)/\sigma) = F_Z((x-\mu)/\sigma)$

e. Denote $F_Z(z)$ by $\Phi(z)$, the standard normal cdf.

f. So $F_X(x) = \Phi((z-\mu)/\sigma)$.

g. Φ is tabulated in book.

: 5.2

6. Linear combination of two independent standard normal random variables is normal

   a. Suppose $X \sim \mathsf{N}(0,1)$, $Y \sim \mathsf{N}(0,1)$, $X \perp Y$, $Z = aX + bY$ for $b > 0$, $a \neq 0$. Then $Z \sim \mathsf{N}(0, a^2 + b^2)$.

      i. Integrate over region with $x \in (-\infty, \infty)$, $y \in (-\infty, (z - ax)/b)$:

$$P(Z \le z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{z-ax}{b}} \frac{\exp(-x^2/2 - y^2/2)}{2\pi} \, dy \, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z} \frac{\exp(-x^2/2 - (\frac{w-ax}{b})^2/2)}{2\pi} \, b^{-1} dw \, dx$$

using $y = (w - ax)/b$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z} \frac{\exp(-\frac{v^2+w^2}{2(a^2+b^2)})}{2\pi} \, b^{-1} \frac{b}{a^2+b^2} dw \, dv$$

using $x = (aw + bv)/(a^2 + b^2)$

$$= \int_{-\infty}^{z} \frac{\exp(-\frac{w^2}{2(a^2+b^2)}}{\sqrt{2\pi})} \frac{1}{\sqrt{a^2+b^2}} dw$$

7. Linear combination of any number of independent general normal random variables is normal

   a. Recall that means and variances both add.

   b. So if X_i are independent $\mathsf{N}(\mu, \sigma^2)$ then $\bar{X} = \Sigma_{i=1}^{n} X_i/n \sim$

$\mathsf{N}(\mu, \sigma^2/n)$.