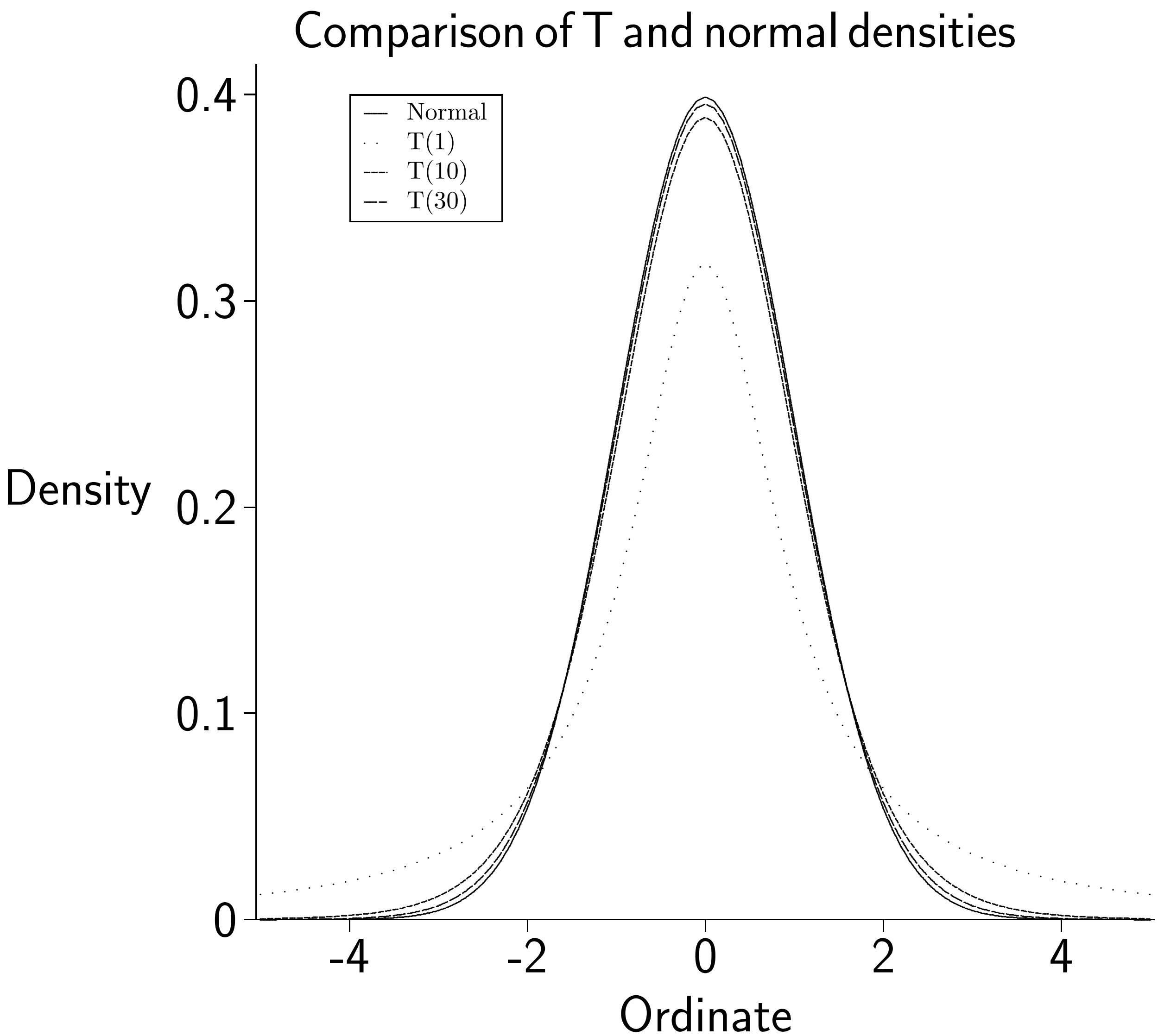
: 5.4.3

2. Recall that if $X_1, \ldots, X_n \sim \mathsf{N}(\mu, \sigma^2)$ and the observations are independent then $(\bar{X} - \mu)/(\sigma/\sqrt{n}) \sim \mathsf{N}(0,1)$
3. If we don't know σ , can we replace it by its estimator s and compare to $\mathsf{N}(0,1)$?
4. ie., what is the distribution of $T = (\bar{X} - \mu)/(s/\sqrt{n}) = ((\bar{X} - \mu)/\sigma)/(s/(\sigma\sqrt{n}))$?
5. μ and σ wash out.
6. Probabilities can be calculated by integral of product of normal and χ^2 density.
   a. T has same distribution as $Z/\sqrt{C/(n-1)}$ for $C \sim \chi^2_{n-1}$.
   b. For $t \ge 0$, $P(T \ge t) = \frac{1}{2}P\left(T^2 \ge t^2\right) = \frac{1}{2}P\left(Z^2(n-1) \ge t^2 C\right)$
   c. Let $U = Z^2$ with density $\exp(-u/2)(1/2)^{1/2}u^{-1/2}/\sqrt{\pi}$.
   d. For $t \ge 0$,
   $$\begin{aligned} P(T \ge t) &= \tfrac{1}{2}P\left(U(n-1) \ge t^2 C\right) \\ &= \tfrac{1}{2}\int_0^\infty \int_{ct^2/(n-1)}^\infty f_U(u) f_C(c)\, du\, dc \end{aligned}$$
7. Density is $\frac{1}{2}\int_0^\infty f_U(ct^2/(n-1)) f_C(c)(2tc/(n-1))\, du\, dc$

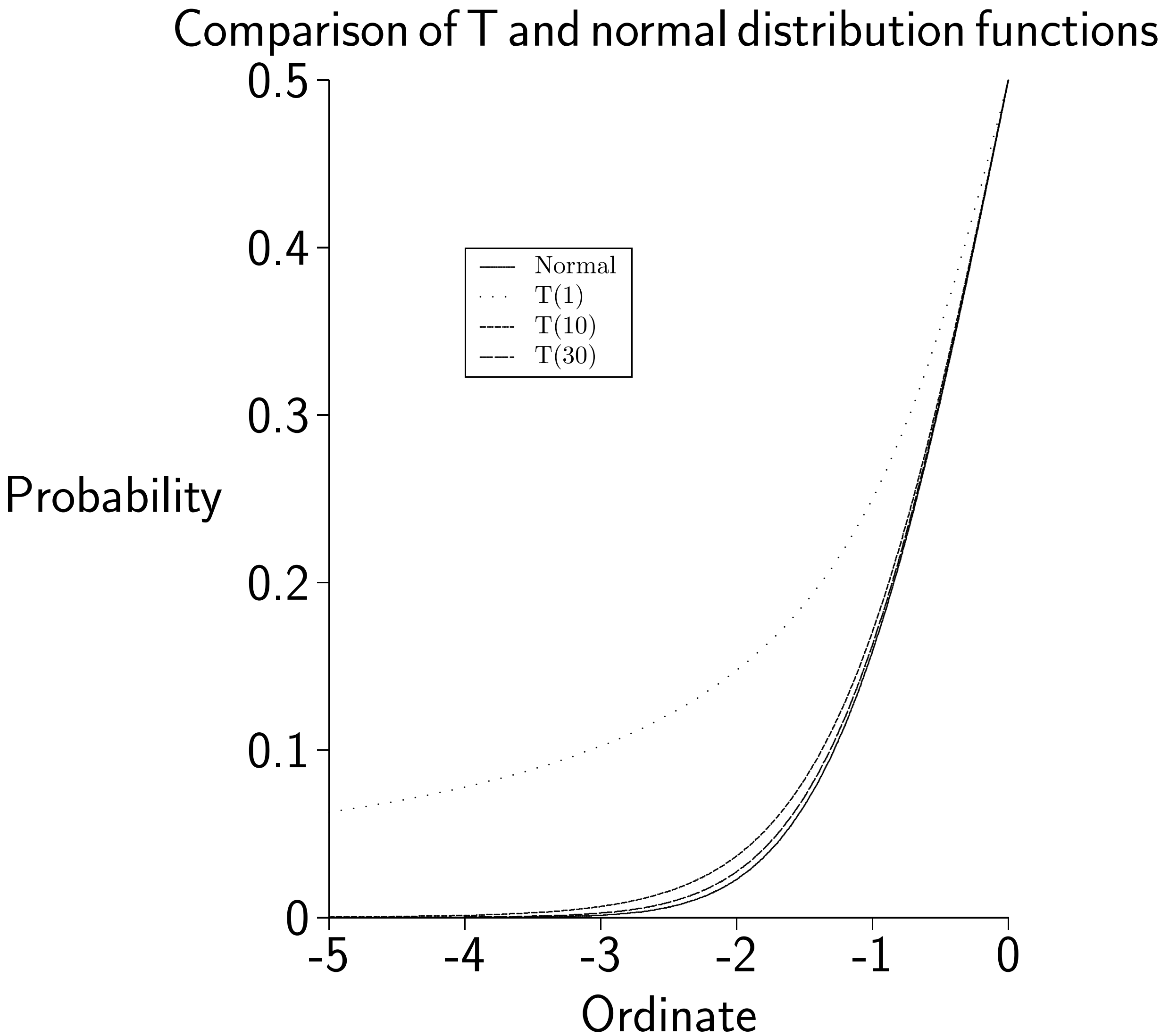
a. Resulting integral is a gamma integral, times a constant in c.

b. Result is $f_T(t) \propto (1+t^2/(n-1))^{n/2}$



c. When $n = 2$ we get the Cauchy density providing the counterexample to the CLT

8. Result is called t_{n-1}, read "t distribution on $n-1$ degrees of freedom".



9. CDF accessible by `pt(x,n-1)` in R.
10. For $n \geq 40$, result is indistinguishable from standard normal.

F. Describing estimators

1. The difference between the expectation of the estimator and the quantity that it is estimating is called the $bias$. glossa
   a. Formally, bias of an estimator T of a parameter θ is $\mathrm{E}[T] - \theta$.
2. An estimate of the standard deviation of an estimator is called its

standard error .

glossa stand

3. Expected difference between estimator and parameter is called *mean square error* .

glossa square

   a. Combines ideas of bias and variance
   b. MSE$= \mathrm{E}[(T-\theta)^2] = \mathrm{E}[(T-\mathrm{E}[T])^2] + \mathrm{E}[(\mathrm{E}[T]-\theta)^2] + \mathrm{E}[(\mathrm{E}[T]-\theta)(T-\mathrm{E}[T])] =$ Var $[T] + (\mathrm{E}[T]-\theta)^2 + 0 =$ bias squared plus variance

: 7.4

G. Given a probability model for data with one or more unknown parameters, how do we construct a good estimator

1. Method of Moments
   a. Assume observations are iid.
   b. Define equations equating sample moments with population moments
      i. If you have one parameter, solve $\bar{X} = \mathrm{E}[X_j]$
      ii. If you have two parameters, solve previous and $\Sigma_{j=1}^{n} X_j^2/n = \mathrm{E}[X_j^2]$
   - Same as $\bar{X} = \mathrm{E}[X_j]$ and $\Sigma_{j=1}^{n}(X_j - \bar{X})^2/n =$ Var $[X_j]$.
   - Similar to $\bar{X} = \mathrm{E}[X_j]$ and $\Sigma_{j=1}^{n}(X_j - \bar{X})^2/(n-1) =$

Var $[X_j]$, which the book sugests. Let's do this one.

c. Examples

i. Ex. $X_j \sim$ Pois(λ)

- Estimator satisfies $\bar{X} = \hat{\lambda}$.
- It's conventional to denote an estimator for the parameter by the symbol for the estimator with a hat on it.

ii. Ex. $X_j \sim \Gamma(k, \lambda)$, for k known

- Estimator satisfies $\bar{X} = k/\hat{\lambda}$, or $\hat{\lambda} = k/\bar{X}$. and $\Sigma_{j=1}^{n} X_j^2/n = \mathrm{E}[X_j^2]$.

iii. Ex. $X_j \sim \Gamma(k, \lambda)$

- Estimator satisfies $\bar{X} = \hat{k}/\hat{\lambda}$, $s^2 = \hat{k}/\hat{\lambda}^2$.
- $\hat{\lambda} = \bar{X}/s^2$; $\hat{k} = (\bar{X})^2/s^2$.

iv. Ex. $X_j \sim$ N(μ, σ^2)

- Estimator satisfies $\bar{X} = \hat{\mu}$, $s^2 = \hat{\sigma}^2$.

v. Ex. $X_j \sim$ U$(0, \theta)$

- $\hat{\theta}$ satisfies $\theta/2 = \bar{X}$, or $\theta = 2\bar{X}$.
- Estimator is unbiased: $\mathrm{E}[2\bar{X}] = \theta$