960-381– Theory of Probability– Fall, 2021

Exam 1

You have until 12:20 PM to finish this exam.

1. (20 pts) (From DeGroot (1989) p. 67) Three different machines were used to produce a large batch of similar manufactured items. Of these items, .2 come from machine 1, .3 come from machine 2, and .5 come from machine 3. Furthermore, .01 of those items produced by machine 1 are defective, and .02 of those items produced by machine 2 are defective, and .03 of those items produced by machine 3 are defective. Suppose that one of this type of item is selected at random. Conditional on this item being defective, find the probabilities that the item came from machine 1, 2, or 3.

Use Bayes' Rule:

$$P(\textit{machine } 1|\textit{defective})$$
$$= \frac{P(\textit{defective}|\textit{machine } 1)\, P(\textit{machine } 1)}{\sum_{j=1}^{3} P(\textit{defective}|\textit{machine } j)\, P(\textit{machine } j)}$$
$$= \frac{.01 \times .2}{.01 \times .2 + .02 \times .3 + .03 \times .5}$$
$$= \frac{2}{23} = .087$$
$$P(\textit{machine } 2|\textit{defective}) = \frac{.02 \times .3}{.01 \times .2 + .02 \times .3 + .03 \times .5}$$
$$= \frac{6}{23} = .261$$
$$P(\textit{machine } 3|\textit{defective}) = \frac{.03 \times .5}{.01 \times .2 + .02 \times .3 + .03 \times .5}$$
$$= \frac{15}{23} = .652$$

2. Consider a collection of six cards, numbered 1 through 6. Supposed that the cards are randomly rearranged so that all rearrangements are equally likely. Let X be the sum of the first two cards in the deck.

a. (10 pts) Give the probability distribution for X .

The cells in this table are all equally likely:

	1	*2*	*3*	*4*	*5*	*6*
1		*3*	*4*	*5*	*6*	*7*
2	*3*		*5*	*6*	*7*	*8*
3	*4*	*5*		*7*	*8*	*9*
4	*5*	*6*	*7*		*9*	*10*
5	*6*	*7*	*8*	*9*		*11*
6	*7*	*8*	*9*	*10*	*11*	

Then the sums have the following p.m.f.:

x	3	4	5	6	7	8	9	10	11
p	1/15	1/15	2/15	2/15	1/5	2/15	2/15	1/15	1/15

b. (10 pts)Calculate $\mathrm{E}(X)$.

$$E(X) = 3 \times 1/15 + 4 \times 1/15 + 5 \times 2/15 + 6 \times 2/15 + 7 \times 3/15$$
$$+ 8 \times 2/15 + 9 \times 2/15 + 10 \times 1/15 + 11 \times 1/15 = 7.$$

c. (10 pts)Calculate $\mathrm{V}(X)$.

$$E\left(X^2\right) = 3^2 \times 1/15 + 4^2 \times 1/15 + 5^2 \times 2/15 + 6^2 \times 2/15 +$$
$$7^2 \times 3/15 + 8^2 \times 2/15 + 9^2 \times 2/15 + 10^2 \times 1/15 +$$
$$11^2 \times 1/15 = 53.66667.$$
$$V(X) = E\left(X^2\right) - E(X)^2 = 53.667 - 49 = 4.667.$$

3. (20 pts) (From DeGroot (1989) p. 55) Supposed that a die is independently rolled 3 times, and that X_i represents the outcome from roll i . Find $\mathrm{P}(X_1 > X_2 > X_3)$.

Let $A = \{X_1 \neq X_2, x_1 \neq X_3, X_2 \neq X_3\}$ *, and* $B = \{X_1 > X_2 > X_3\}$ *. Note that* $P(A) = (5/6)(2/3) = 5/9$ *,* $P(B|A) = 1/6$ *, and* $P(B|A^c) = 0$ *. Then* $P(B) = (1/6)(5/9) = 5/54$ *. The brute-force way to to this is to note that the following sequences of rolls satisfy the event:*

X_1	X_2	X_3	*Number*	X_1	X_2	X_3	*Number*
6	*5*	*1-4*	*4*	*6*	*4*	*1-3*	*3*
6	*3*	*1-2*	*2*	*6*	*2*	*1*	*1*
5	*4*	*1-3*	*3*	*5*	*3*	*1-2*	*2*
5	*2*	*1*	*1*	*4*	*3*	*1-2*	*2*
4	*2*	*1*	*1*	*3*	*2*	*1*	*1*

There are 20 such sets, and $6^3 = 216$ *tables total, for a probability* $20/2016 = 5/54$ *. One student noted that the number of tables satisfying this requirement is* $\binom{6}{3}$ *, since every set of ordered dice rolls corresponds to one selection of three items from* $\{1, \ldots, 6\}$ *.*

4. I buy a carton of toothbrushes. The carton has 20 tooth brushes. Three of them are blue. I pull out four tooth brushes at random, one for each member of my family. Let X represent the number of toothbrushes that my family uses that are blue.

a. (10 pts) What is the expectation of the distribution of X ?

$E(X) = 3 \times 4/20 = .6$.

b. (10 pts) Calculate $\mathrm{P}(X \geq 2)$.

$P(X \geq 2) = P(X = 2) + P(X = 3) = \frac{\binom{3}{2}\binom{17}{2}}{\binom{20}{4}} + \frac{\binom{3}{3}\binom{17}{1}}{\binom{20}{4}} = \frac{3\times17\times8}{5\times19\times3\times17} + \frac{17}{5\times19\times3\times17} =$ $425/4845 = 5/57 = 0.0877$.

c. (10 pts) What term is given for the sampling scheme given above?

Sampling without replacement.