**960-381– Theory of Probability– Fall, 2021**

## Homework 2 Solutions, 04 Oct

1. Question 3.6, page 90. Five balls numbered 1,2,3,4,5 are placed in an urn. Two balls are randomly selected from the five, and their numbers noted. Find the probability distributions for the following.

a. The largest of the two sampled balls.

*The table below is a table of possible first two ball draws, and the maximum.*

| *First Ball* | *Second Ball* | *Maximum* | *First Ball* | *Second Ball* | *Maximum* |
|---|---|---|---|---|---|
| *2* | *1* | *2* | *4* | *3* | *4* |
| *3* | *1* | *3* | *5* | *3* | *5* |
| *4* | *1* | *4* | *1* | *4* | *4* |
| *5* | *1* | *5* | *2* | *4* | *4* |
| *1* | *2* | *2* | *3* | *4* | *4* |
| *3* | *2* | *3* | *5* | *4* | *5* |
| *4* | *2* | *4* | *1* | *5* | *5* |
| *5* | *2* | *5* | *2* | *5* | *5* |
| *1* | *3* | *3* | *3* | *5* | *5* |
| *2* | *3* | *3* | *4* | *5* | *5* |

*There are 20 such combinations, giving duplicate maxima with various frequencies. These are given here:*

| *Maximum* | *2* | *3* | *4* | *5* |
|---|---|---|---|---|
| *Count* | *2* | *4* | *6* | *8* |
| *Probability* | *.1* | *.2* | *.3* | *.4* |

*Probabilities are given by dividing these counts by 20, and are given by the final row in the table above.*

b. The sum of the two sampled balls.

| *First Ball* | *Second Ball* | *Sum* | *First Ball* | *Second Ball* | *Sum* |
|---|---|---|---|---|---|
| *2* | *1* | *3* | *4* | *3* | *7* |
| *3* | *1* | *4* | *5* | *3* | *8* |
| *4* | *1* | *5* | *1* | *4* | *5* |
| *5* | *1* | *6* | *2* | *4* | *6* |
| *1* | *2* | *3* | *3* | *4* | *7* |
| *3* | *2* | *5* | *5* | *4* | *9* |
| *4* | *2* | *6* | *1* | *5* | *6* |
| *5* | *2* | *7* | *2* | *5* | *7* |
| *1* | *3* | *4* | *3* | *5* | *8* |
| *2* | *3* | *5* | *4* | *5* | *9* |

*These are given here:*

| *Sum* | *3* | *4* | *5* | *6* | *7* | *8* | *9* |
|---|---|---|---|---|---|---|---|
| *Count* | *2* | *2* | *4* | *4* | *4* | *2* | *2* |
| *Probability* | *.1* | *.1* | *.2* | *.2* | *.2* | *.1* | *.1* |

*Probabilities are given by dividing these counts by 20, and are given by the final row in the table above.*

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2. Question 3.26, page 99. A heavy-equipment salesperson can contact either one or two customers per day with the probability 1/3 and 2/3 respectively. Each contact will result in either no sale or a $50,000 sale with the probabilities .9 and .1, respectively. Give the probability distribution for daily sales. Find the mean and standard deviation of the daily sales.

*A salesperson can have 0, 1, or 2 sales. Let* $X$ *represent the number of sales. The probability of 2 sales is the probability of contacting two customers, times the probability of making both sales. Hence* $P(X=2) = 2/3 \times 1/10 \times 1/10 = 1/150$ *. The probability of one sale is the probability of one contact times the probability of making the sale, plus the probability of two contacts times the probability of making the first sale and failing to make the second sale, plus the probability of two contacts times the probability of making the second sale and failing to make the first sale. Hence* $P(X=1) = 1/3 \times 1/10 + 2/3 \times 1/10 \times 9/10 \times 2 = 1/30 + 3/25 = 23/150$ *. As a check, I also check the probability of zero sales. This is the probability of 1 contact times the probability of failing to make the sale, plus the probability of two contacts, times the probability of failing both times. Hence* $P(X=0) = 1/3 \times 9/10 + 2/3 \times 9/10 \times 9/10 = 3/10 + 27/50 = 42/50$ *. These three probabilities sum to one. The expectation is* $E(X) = 23/150 + 2/150 = 25/150 = 1/6$ *. The expectation of the square is* $23/150 + 4/150 = 27/150$ *. The variance is* $27/150 - 1/6 = 1/75$ *, and the standard deviation is* $1/(5\sqrt{3})$ *. Hence the expectation and standard deviation of daily sales are the corresponding values for* $X$ *, times £50,000, and so expectation and standard deviation of dollar value of sales is £50,000/6=£8333 and* $\$50,000 \times 1/(5\sqrt{3}) = \$5773$ *.*

3. Question 3.40, page 111. The probability that a patient recovers from a stomach disease is .8. Suppose 20 people are known to have contracted this disease. What is the probability that

a. exactly 14 recover?

$\binom{20}{14}.8^{14}.2^{6} = 0.1091$ . *You can do this with* `dbinom(14,20,.8)`*.*

b. at least 10 recover?

*One can add probabilities like that in the previous part. Otherwise, apply* `1-pbinom(9,20,.8)` *to get 0.9994.*

c. at least 14 but not more than 18 recover?

*Once can add probabilities for 14, 15, 16, 17, and 18, perhaps via* `pbinom(18,20,.8)-pbinom(13,20,.8)` *to get 0.8441.*

d. at most 16 recover?

Do it via `pbinom(16,20,.8)` to get 0.5886.

4. Question 3.62, page 114. Granson and Hall (1980) explain that the probability of detecting a crack in an airplane wing is the product of $p_1$ , the probability of inspecting a plane with a wing crack, and $p_2$ , the probability of inspecting the detail in which the crack is located; and $p_3$ , the probability of detecting the damage.

a. What assumptions justify the multiplications of these properties?

*These probabilities multiply if they can be interpreted conditionally; that is,* $p_2$ *needs to be the probability of inspecting a wing conditional on the presence of a crack, and* $p_3$ *is the probability of detecting damage given that the area was selected for inspection. The probability* $p_3$ *does not allow for an alternative interpretation, but the second event, that of inspection, might be dependent on the presence of the crack.*

b. Suppose $p_1 = .9$ , $p_2 = .8$ and $p_3 = .5$ for a certain fleet of planes. If three planes are inspected from this fleet, find the probability that a wing crack will be detected on at least one of them.

*The result is* $.9 \times .8 \times .5 = .36$ .

5. Question 3.70, page 119. An oil prospector will drill a succession of holes in a given area to find a productive well. The probability that he is successful in a given trial is .2.

a. What is the probability that the third hole drilled is the first to yield a productive well?

*The probability is* $.8 \times .8 \times .2 = .128$ .

b. If the prospector can afford to drill at most ten wells, what is the probability that he will fail to find a productive well?

*The probability is* $.8^{10} = .107$ .

6. Question 3.96, page 124. The telephone lines serving an airline reservation office are all busy about 60% of the time.

a. If you are calling this office, what is the probability that you will complete your call on the first try? The second try? The third try?

*The probability of success on the first try is* $.4$ . *The probability of first success on the second try is* $.6 \times .4 = .24$ . *The probability of first success on the third try is* $.6 \times .6 \times .4 = .144$ .

b. If you and your friend both complete calls to this office, what is the probability that a total of four tries will be necessary for both of you to get through?

*Four tries can be spilt evenly, with probability* $.24$ *each, gives probability* $.24^2$ . *Or, one person can make three tries, and one can make one try, each with probability* $.4 \times .144$ , *and there are two ways to do this, for an overall probability of* $2 \times .4 \times .144 = 0.1152$ . *Then the probability of four calls is* $0.1152 + .24 = 0.3552$ .

7. Question 3.104, page 128. Twenty identical-looking packages of white powder are such that 15 contain cocaine and 5 do not. Four packets were randomly selected, and the contents were tested and found to contain cocaine. Two additional packets were selected from the remainder and sold by undercover police officers to a single buyer. What is the probability that the 6 packets randomly selected are such that the first 4 all contain cocaine and the 2 sold to the buyer do not?

*Treat these as hypergeometric:*

| | | |
|---|---|---|
| *X* | | *15* |
| | | *5* |
| *6* | *14* | *20* |

*The probability that* $X = 4$ *is*

$$\frac{\binom{15}{4}\binom{5}{2}}{\binom{20}{6}} = 0.352.$$

8. Question 3.136, page 137. Increased research and discussion have focused on the number of illnesses involving the organism *Escherichia coli* (10257:H7), which causes a breakdown of red blood cells and intestinal hemorrhages in its victim. Sporadic outbreaks of *E. coli* have occurred in Colorado at a rate of approximately 2.4 per 100,00 for a period of two years.

a. If this rate has not changed and if 100,000 cases from Colorado are reviewed for this year, what is the probability that at least 5 cases of *E. coli* will be observed.

*This can be treated as a binomial count from 100,000 trials and a success probability of* 2.4/100000 *;* `1-pbinom(4,100000,2.4/100000)` *gives 0.0959. Alternatively, one might model this as Poisson; the probability is* `1-ppois(4,2.4)`*, which gives the same result to three decimal places.*

b. If 100,000 cases from Colorado are reviewed for this year and the number of *E. coli* exceeds 5, would you suspect that the state's mean *E. coli* rate has changed? Explain?

*If the population rate remained stable, a number of cases this high would be very unusual; one might reject the hypothesis that the rate is constant.*

9. Question 3.156, page 142. Suppose that $Y$ is a random variable with moment generating function $m(t)$ .

a. What is $m(0)$ ?

$m(0) = E(\exp(Y \times 0)) = E(1)$ .

b. If $W = 3Y$ , show that the moment generating function of $W$ is $m(3t)$ .

$m_W(t) = E(\exp(tW)) = E(\exp(t3Y)) = m(3t)$ .

c. If $X = Y - 2$ , show that the moment generating function of $X$ is $e^{-2t}m(t)$ .

$m_X(t) = E(\exp(tX)) = E(\exp(t(Y - 2))) = \exp(-2t)m(t)$ .