960-381– Theory of Probability– Fall, 2021

Homework 5 Solutions, 18 Nov

1. Question 5.38, page 246. Let Y_1 denote the weight (in tons) of a bulk item stocked by a supplier at the beginning of a week and suppose that Y_1 has a uniform distribution over the interval $0 \leq y_1 \leq 1$. Let Y_2 denote the amount (by weight) of this item sold by the supplier during the week and suppose that Y_2 has a uniform distribution over the interval $0 \leq y_2 \leq y_1$, where y_1 is a specified value of Y_1 .

a. Find the joint density function for Y_1 and Y_2 .

Note that

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2|Y_1}(y_2|y_1).$$

Since $f_{Y_1}(y_1) = 1$ *and* $f_{Y_2|Y_1}(y_2|y_1) = \frac{1}{y_1}$, *the joint density is*

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 1/y_1 & \text{if } 0 < y_2 < y_1 < 1 \\ 0 & \text{otherwise} \end{cases}.$$

b. If a supplier stocks a half-ton of the item, what is the probability that she sells more than a quarter-ton?

$(1/2 - 1/4)/(1/2) = 1/2$.

c. If it is known that the suppliers sold a quarter-ton of the item, what is the probability that she had stocked more than a half-ton?

$f_{Y_2}(y_2) = \int_{y_2}^{1} y_1^{-1}\, dy_1 = -\ln(y_2)$ *for* $y \in (0, 1)$. *Then* $f_{Y_1|Y_2}(y_1|y_2) = \begin{cases} -1/(y_1 \ln(y_2)) & \text{if } 1 > y_1 > y_2 \\ 0 & \text{other} \end{cases}.$

2. Question 5.65, page 254. Suppose that, for $-1 \leq \alpha \leq 1$, the probability density function of (Y_1, Y_2) is given by

$$f(y_1, y_2) = \begin{cases} [1 - \alpha\{(1 - 2\exp(-y_1))(1 - 2\exp(-y_2))\}]\exp(-y_1 - y_2) & 0 \leq y_1, 0 \leq y_2 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Show that the marginal distribution of Y_1 is exponential with mean 1.

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty [1 - \alpha\{(1 - 2\exp(-y_1))(1 - 2\exp(-y_2))\}]\exp(-y_1 - y_2)]\, dy_2 \\ &= (1 - \alpha)\exp(-y_1) \\ &+ \alpha \int_0^\infty [2\exp(-2y_1 - y_2) + 2\exp(-2y_2 - y_1) - 4\exp(-2y_1 - 2y_2)]\, dy_2 \\ &= (1 - \alpha)\exp(-y_1) + \alpha[2\exp(-2y_1) + \exp(-y_1) - 2\exp(-2y_1)] \\ &= (1 - \alpha)\exp(-y_1) + \alpha[\exp(-y_1)] \\ &= \exp(-y_1). \end{aligned}$$

b. What is the marginal distribution of Y_2 ?

By symmetry, the distribution of Y_2 *is also exponential with mean 1.*

c. Show that Y_1 and Y_2 are independent if and only if $\alpha = 0$.

If $\alpha = 0$ *, then the joint density is* $\exp(-y_1)\exp(-y_2)$ *. Since this is the product of functions of the arguments representing random variables, these random variables are independent. If the two variables are independent, then* $f_{Y_2|Y_1}(y_2|y_1)$ *must not depend on* y_1 *. But* $f_{Y_2|Y_1}(y_2|y_1) = [1 - \alpha\{(1-2\exp(-y_2))(1-2\exp(-y_2))\}]\exp(-y_2)$ *. This depends on* y_1 *unless* $\alpha = 0$ *.*

Observing that $E(Y_1Y_2) \neq E(Y_1)\,E(Y_2)$ *if* $\alpha \neq 0$ *proves that independence requires* $\alpha = 0$ *, but observing that* $E(Y_1Y_2) = E(Y_1)\,E(Y_2)$ *if* $\alpha = 0$ *is not sufficient to prove that* $\alpha = 0$ *guarantees independence.*

3. Question 5.82, page 263. In exercise 5.38, we determined that the joint density function for Y_1 , the weight in tons of a bulk item stocked by a supplier, and Y_2 , the weight of the item sold by the supplier, has a joint density

$$f(y_1, y_2) = \begin{cases} 1/y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

In this case, the random variable $Y_1 - Y_2$ represents the amount of stock remaining at the end of the week, a quantity of great importance to the supplier. Find $\mathrm{E}(Y_1 - Y_2)$.

$$\int_0^1 \int_0^{y_1} (y_1 - y_2)/y_1 \, dy_2 \, dy_1 = \int_0^1 (y_1 y_2 - y_2^2/2)/y_1 \Big|_0^{y_1} \, dy_1 = \int_0^1 (y_1/2) \, dy_1 = 1/2.$$

4. Question 5.114, page 278. For the daily output of an industrial operation, let Y_1 denote the amount of sales and Y_2 denote the cost, in thousands of dollars. Assume the density functions for Y_1 and Y_2 are given by

$$f_1(y_1) = \begin{cases} (1/6)y_1^3\exp(-y_1), & y_1 > 0 \\ 0 & y_1 \le 0 \end{cases}$$

and

$$f_2(y_2) = \begin{cases} (1/2)\exp(-y_2/2), & y_1 > 0 \\ 0 & y_1 \le 0. \end{cases}$$

The daily profits are given by $U = Y_1 - Y_2$.

a. Find $\mathrm{E}(U)$.

In the Wackerly notation, $Y_1 \sim \Gamma(4,1)$ *and* $Y_2 \sim \Gamma(1,2)$ *. Hence* $E(Y_1) = 4$ *and* $E(Y_2) = 2$ *. Hence* $E(U) = E(Y_1) - E(Y_2) = 4 - 2 = 2$ *.*

b. Assuming that Y_1 and Y_2 are independent, find $\mathrm{V}(U)$.

In the Wackerly notation, $Y_1 \sim \Gamma(4,1)$ *and* $Y_2 \sim \Gamma(1,2)$ *. Hence* $V(Y_1) = 4$ *and* $V(Y_2) = 4$ *. By independence, variances add. Hence* $V(U) = V(Y_1) + V(Y_2) = 4 + 4 = 8$ *.*

c. Would you expect the daily profit to drop below zero very often? Why?

Yes. $P(Y_2 > Y_1) = \int_0^\infty \int_0^{y_2} (1/6) y_1^3 \exp(-y_1)(1/2)\exp(-y_2/2)\; dy_1\, dy_2 = 16/81$.

5. Question 5.123, page 283. The National Fire Incident Reporting Service stated that, among residential fires, 73% are in family homes, 20% are in apartments, and 7% are in other types of dwellings. If four residential fires are independently reported on a single day, what is the probability that two are in family homes, one is in an apartment, and one is in another type of dwelling?

$\frac{4!}{2!1!1!} 0.73^2 \times 0.20 \times 0.07 = 12 \times 0.73^2 \times 0.20 \times 0.07 = 0.0895$.

6. Question 5.142, page 290. Suppose that Y has a binomial distribution with parameters n and p but that p varies from day to day according to a beta distribution with parameters α and β . Show that

a. $\mathrm{E}(Y) = \frac{n\alpha}{\alpha+\beta}$. Here use $\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|p)) = \mathrm{E}(np) = n\mathrm{E}(p) = n\alpha/(\alpha+\beta)$.

b. $\mathrm{V}(Y) = \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

Here use $V(Y) = V(E(Y|p)) + E(V(Y|p)) = V(np) + E(np(1-p)) = n V(p) + nE(p) - nE(p^2) = n\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + n\alpha/(\alpha+\beta) - n\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$.

7. Question 6.31, page 317. The joint distribution for the length of life of two different types of components operating in a system was given in Exercise 5.18 by

$$f(y_1, y_2) = \begin{cases} (1/8) y_1 \exp(-(y_1+y_2)/2), & y_1 > 0, y_2 > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

The relative efficiency of the two types of components is measured by $U = Y_2/Y_1$. Find the probability density function for U .

Begin by integrating the joint density over the set $U \le u$. *Then* $F_U(u) = \int_0^\infty \int_0^{uy_1} (1/8) y_1 \exp(-(y_1+y_2)/2)\; dy_2\, dy_1 = \int_0^\infty \int_0^{uy_1/2} (1/4) y_1 \exp(-y_1/2)\exp(-x)\; dx\, dy_1 = \int_0^\infty (1/4) y_1 \exp(-y_1/2)(1-\exp(-uy_1/2))\; dy_1 = \int_0^\infty (1/4) y_1 (\exp(-y_1/2) - \exp(-(1+u)y_1/2)\; dy_1 = 1 - 1/(1+u)^2$. *Then* $f_U(u) = \frac{d}{du}(1 - 1/(1+u)^2) = 2/(1+u)^3$.

Alternatively, use the transformation formula. Let $Z = Y_1$. *Then* $Y_1 = Z$ *and* $Y_2 = Y_1 U = UZ$. *Then* $f_{U,Z}(u,z) = f_{Y_1,Y_2}(z, uz) J^-$ *where* $J^- = |\begin{pmatrix} \partial y_1/\partial u & \partial y_1/\partial z \\ \partial y_2/\partial u & \partial y_2/\partial z \end{pmatrix}| = |\begin{pmatrix} 0 & 1 \\ z & u \end{pmatrix}| = z$. *Then* $f_{U,Z}(u,z) = (1/8) z^2 \exp(-z(1+u)/2)$, *and* $f_U(u) = \int_0^\infty (1/8) z^2 \exp(-z(1+u)/2)\; dz = \int_0^\infty (1+u)^{-3} v^2 \exp(-v)\; dv = 2(1+u)^{-3}$.