WMS: 3.4

E. Particular Distributions

1. Binomial Distribution:

a. observe m trials, each of which could be success or failure.

$Binomial\ variable$ is one whose distribution is number of S .

i. all independent

ii. all with same probability π of S (hand hence probability $1-\pi$ of F).

b. Example:

i. Identical twins reared by different parents, with one set of parents mentally ill.

ii. Success is that child with ill parents is just as healthy as child with well parents

iii. If environment has no effect, S should happen half of the time.

c. How many strings of S and F are there?

i. If the m items are identifiable, there are $m! = m \times (m-1) \times (m-2) \times \cdots \times 2 \times 1$ ways to order labels for S and F .

ii. Not all of these orderings are unique.

- each legitimate ordering of S is $permuted$ $x!$ times, and
- each legitimate ordering of F is permuted $(m-x)!$ times.

iii. In order to have this notation make sense when $x = 0$, define $x! = 1$.

iv. Hence there are $\binom{m}{x} = \frac{m!}{x!(m-x)!}$ orderings of S and F giving rise to the same total number of S.

- This is the way of choosing x items from m items without regard to the order of selection, or the number of $combinations$ of x items form m items.

d. P $(X = x) = \binom{m}{x}\pi^x(1-\pi)^{m-x}$.

i. Sample space is the set of all sequences of S and F of length m.

ii. Many sequences of S and F give the same total number of S: $SFSF$, $FFSS$, $FSSF$ for example.

iii. The probability of any one of these strings is the same: $\pi^x(1-\pi)^{m-x}$.

iv. Since sample space is finite, probability of the event $X = x$ is sum of all strings with x S

v. Equals probability of one string times number of strings giving

x S.

- $\binom{m}{x}$ is called a *binomial coefficient*

vi. Do these things sum to 1?

- Note that $1 = (\pi + (1 - \pi))^m = \sum_{j=0}^{m} \binom{m}{j} \pi^j (1 - \pi)^{m-j}$: The binomial theorem.

2. Denote the distribution by Bin(n, π)

a. Example:

i. Urn contains M red tickets among a total number of N.

ii. Draw tickets and put the tickets back and reshuffle each time.

iii. $\pi = M/N$,

iv. X is the number of red tickets drawn.

b. Moments: $\mathsf{E}(X) = m\pi$, $\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2$

i. Expectation is $\mathsf{E}(X) = m\pi$, because

- By definition, $\mathsf{E}(X) = \sum_{x=0}^{m} x \binom{m}{x} \pi^x (1 - \pi)^{m-x}$
- Since multiplier x for first term is zero, we can drop the first term: $\mathsf{E}(X) = \sum_{x=1}^{m} \frac{m!}{x!(m-x)!} x \pi^x (1 - \pi)^{m-x}$
- Cancel x in numerator and denominator: $\mathsf{E}(X) = \sum_{x=1}^{m} \frac{m! \pi^x (1-\pi)^{m-x}}{(x-1)!(m-x)!}$
- Adjust to make function of $x - 1$ and $m - 1$:

$\mathsf{E}(X) = \sum_{x=1}^{m} \frac{m(m-1)!\pi\pi^{x-1}(1-\pi)^{m-x}}{(x-1)!((m-1)-(x-1))!}$

- Reindex by $z = x - 1$: $\mathsf{E}(X) = \pi m \sum_{z=0}^{m-1} \frac{(m-1)!}{z!((m-1)-z)!}\pi^z(1-\pi)^{(m-1)-z}$
- Note $\sum_{z=0}^{m-1} \frac{(m-1)!}{z!((m-1)-z)!}\pi^z(1-\pi)^{(m-1)-z} = 1$, because it is the sum of binomial probabilities with $m-1$ trials.

ii. Variance is $m\pi(1-\pi)$, by

- $\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2$
  - ▷ $\mathsf{E}(X(X-1)) + \mathsf{E}(X) - \mathsf{E}(X)^2$
  - ▷ Use cancellation trick to evaluate $\mathsf{E}(X(X-1))$
- or easier trick to follow.

c. Calculation of distribution function:

i. Table 1 from text (Just kidding!)

ii. `pbinom` in R.

3. (Discrete) Uniform Distribution:

a. All probability atoms have the same probability

b. If there are k of them each probability is $1/k$.

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4. Geometric Distribution

a. Observe trials yielding either success or failure (like a coin flip)

i. each with the same probability π of yielding success,

ii. until the first success is observed.

b. Let number of trials needed be random variable N.

c. What is the probability of seeing the first success in the n trial?

i. This can only result from $n-1$ failures followed by one success;

ii. The probability of this is $p_N(n) = (1-\pi)^{n-1}\pi$, as before.

iii. distribution function is $\mathsf{P}(N \leq n) = 1 - \mathsf{P}(N > n) = 1-(1-\pi)^n$.

- The last term is the probability of failure on the first n trials.

iv. $\lim_{n\to\infty} F_N(n) = 1$, as it should.

d. Denote the distribution by Geom(π).

i. Calculate CDF by `pgeom(n-1,p)`.

ii. R definition is number of failures before first success.

- DeGroot text and other sources also use this definition.

e. Expectation $1/\pi$

i. $\sum_{n=1}^{\infty} n p_N(n) = \sum_{n=1}^{\infty} n(1-\pi)^{n-1}\pi$

ii. This infinite sum is not one that you will likely recognize.

iii. We know the result for a geometric series: $\sum_{n=1}^{\infty}(1-\pi)^n =$

$(1-\pi)/\pi$

- Let $Q = \sum_{n=1}^{\infty}(1-\pi)^n$.
- Then $Q = (1-\pi) + \sum_{n=2}^{\infty}(1-\pi)^n = (1-\pi) + (1-\pi)Q$.
- $(\pi - 1 + 1)Q = 1 - \pi$.

iv. Trick: note that summands contain something raised to a power times the power plus 1.

v. Recognize this as as a derivative

- That is, $n(1-\pi)^{n-1} = -\frac{d}{d\pi}(1-\pi)^n$

vi. So $\mathsf{E}(N) = -\pi \sum_{n=1}^{\infty} \frac{d}{d\pi}(1-\pi)^n$

vii. If we can interchange derivative and sum, $\mathsf{E}(N) = -\pi\frac{d}{d\pi}\sum_{n=1}^{\infty}(1-\pi)^n = -\pi\frac{d}{d\pi}(1-\pi)/\pi = -\pi\frac{d}{d\pi}(1/\pi - 1)$

- It is technically easier to justify the interchange if we work backwards, treating it as the integral of a sum equalling the sum of the integrals.

viii. $\mathsf{E}(N) = -\pi(-1/\pi^2) = 1/\pi$

f. Variance $1/\pi^2 - 1/\pi$.

i. Use $\mathsf{V}(N) = \mathsf{E}\left(N^2\right) - \mathsf{E}(N)^2 = \mathsf{E}(N(N-1)) + \mathsf{E}(N) - \mathsf{E}(N)^2$.

ii.

$$\mathsf{E}\left(N(N-1)\right) = \sum_{n=1}^{\infty} n(n-1)p_N(n)$$
$$= \sum_{n=1}^{\infty} n(n-1)(1-\pi)^{n-1}\pi$$

iii. Make this look like second derivative:

$$\mathsf{E}\left(N(N-1)\right) = (1-\pi)\pi \sum_{n=1}^{\infty} n(n-1)(1-\pi)^{n-2}$$
$$= -(1-\pi)\pi \sum_{n=1}^{\infty} \frac{d}{d\pi} n(1-\pi)^{n-1}$$
$$= (1-\pi)\pi \sum_{n=1}^{\infty} \frac{d^2}{d\pi^2}(1-\pi)^{n}$$

iv. As before, can interchange differentiation and summation:

$$\mathsf{E}\left(N(N-1)\right) = (1-\pi)\pi \frac{d^2}{d\pi^2} \sum_{n=1}^{\infty} (1-\pi)^{n}$$
$$= (1-\pi)\pi \frac{d^2}{d\pi^2}(1/\pi - 1)$$
$$= 2(1-\pi)\pi\pi^{-3} = 2(1-\pi)\pi^{-2}$$

v. So: $\mathsf{E}\left(N^2\right) = 2(1-\pi)\pi^{-2} + 1/\pi = 2/\pi^2 - 1/\pi$

vi. So: $\mathsf{V}\left(N\right) = 2/\pi^2 - 1/\pi - 1/\pi^2 = 1/\pi^2 - 1/\pi$

g. Memoryless property of Geometric Tail Probabilities

i. As before, $\mathsf{P}(N > n) = (1-\pi)^n$.

ii. For $y > n$, $\mathsf{P}(N > y | N > n) =$
$\mathsf{P}(\{N > y\} \cap \{N > n\}) / \mathsf{P}(N \geq n) =$
$\mathsf{P}(N > y) / \mathsf{P}(N > n) = (1-\pi)^{y-n}$.

iii. So the distribution of $N - n$, conditional on $N > n$, is Geom(π).
