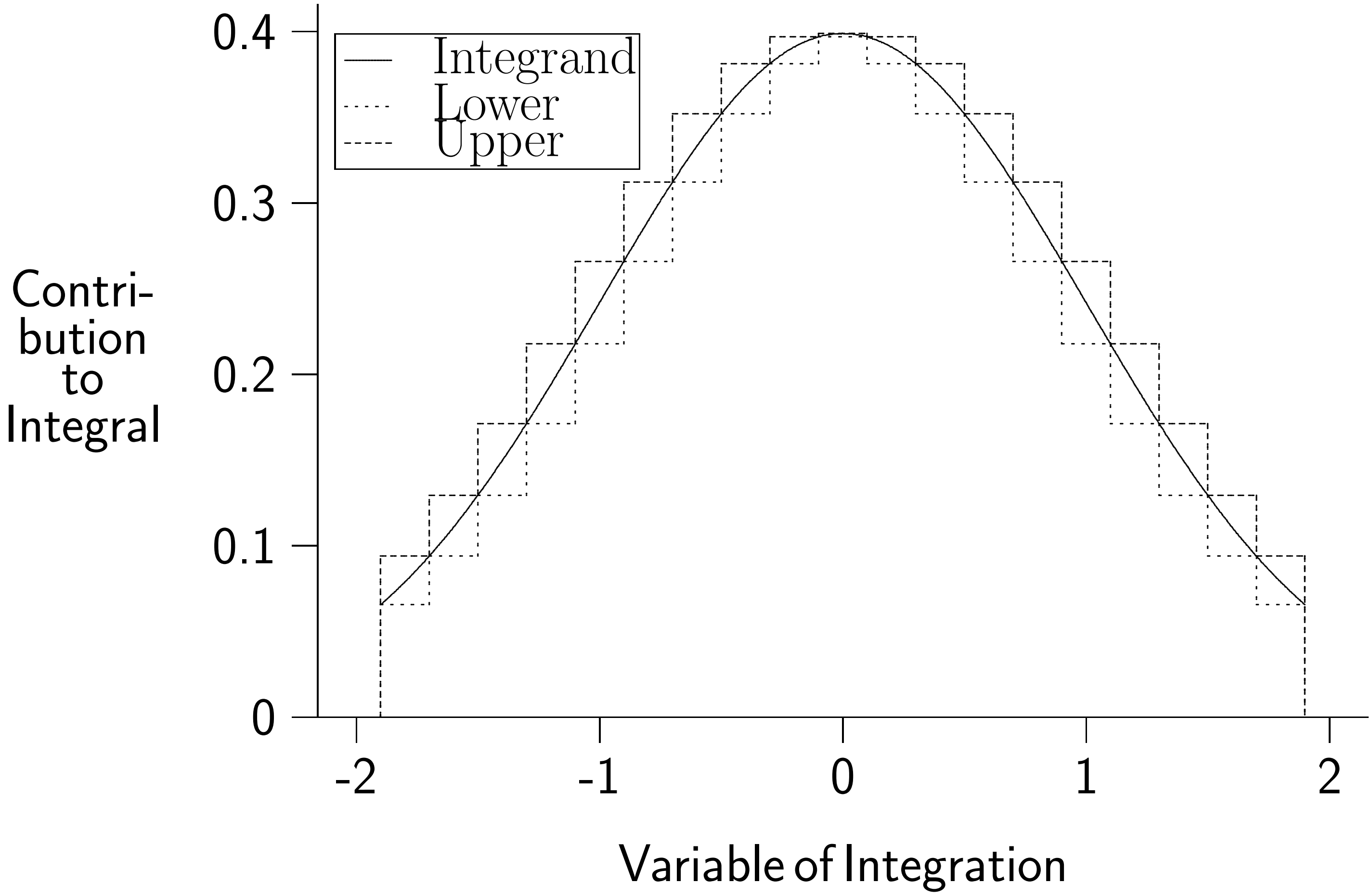
5. Can move $f_X(x)$ at some points without impacting $F_X(x)$.
   a. Recall probability density function is that function whose integral gives probabilities.
   b. $Riemann$ definition of $\int_a^b h(x)\,dx$
      i. Finding sequences of partitions of $[a, b]$
         - as $[a_0, a_1], \ldots, [a_{k-1}, a_k]$ with $a_0 = a$, $a_k = b$
         - so that $\sum_{j=1}^{k}(a_j - a_{j-1}) \max_{x \in [a_{j-1}, a_j]}$ is minimized,

         - so that $\sum_{j=1}^{k}(a_j - a_{j-1}) \min_{x \in [a_{j-1}, a_j]}$ is maximized.
      ii. If upper, lower bounds converge, then common value is the integral. See Fig. 15.
   c. Moving the probability density function at one point induces a change in only one interval.
      i. If this interval shrinks, contribution converges to 0. See Fig. 16.
   d. Hence may ignore values of probability density function at finite number of points.
   e. Hence we will not distinguise between intergrals over ranges like $[a, b]$, $(a, b]$, (a, b), $[a, b)$.

Fig. 15: Lower and Upper Sums for Integration



i. Square bracket indicates end point is included.

ii. Round bracket indicates end point is excluded.

B. Quantile Function as a description of the distribution.

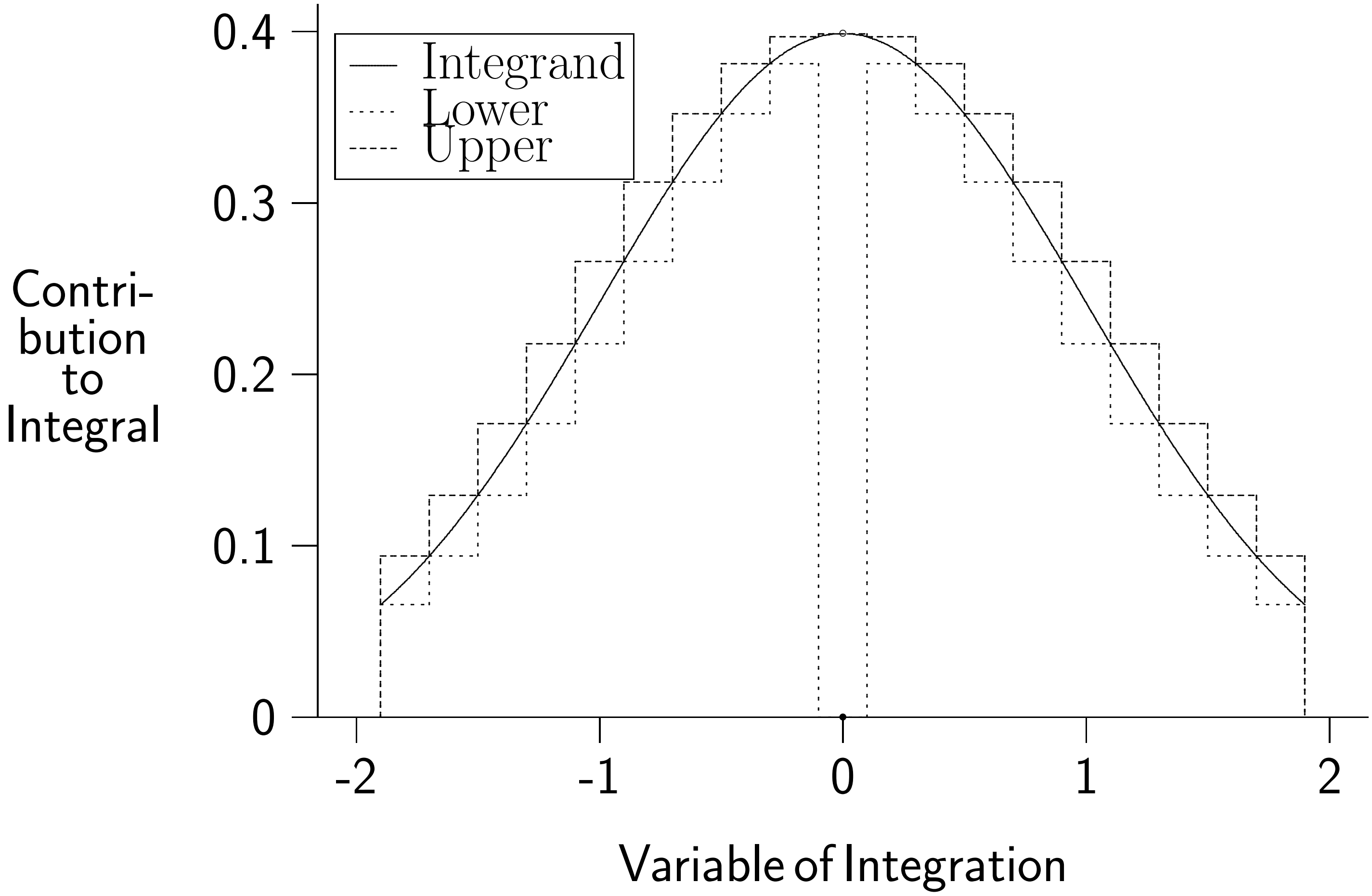
1. Quantile Definition:

a. Heuristically, inverse of $F_X(x)$.

i. See Fig. 17.

b. If unique inverse exists, it is this inverse. See Fig. 18.

Fig. 16: Lower and Upper Sums for Integration



Shows effect of moving integrand to zero at one point

i. Denote the quantile p of random variable X by

$\phi_p = F_X^{-1}(p)$.

c. (At least one) solution to $F_X(\phi_p) = p$ exists if F_X continuous, by intermediate value theorem.

d. Inverse is unique if $f_X(x) > 0$ except at isolated points.

i. Ambiguous if $F_X(x)$ has a flat spot so that various x satisfy $F_X(x) = p$. See Fig. 19.

ii. Then every x value in flat part satisfies quantile function. See

Fig. 17: Easy Distribution with Quantile Function

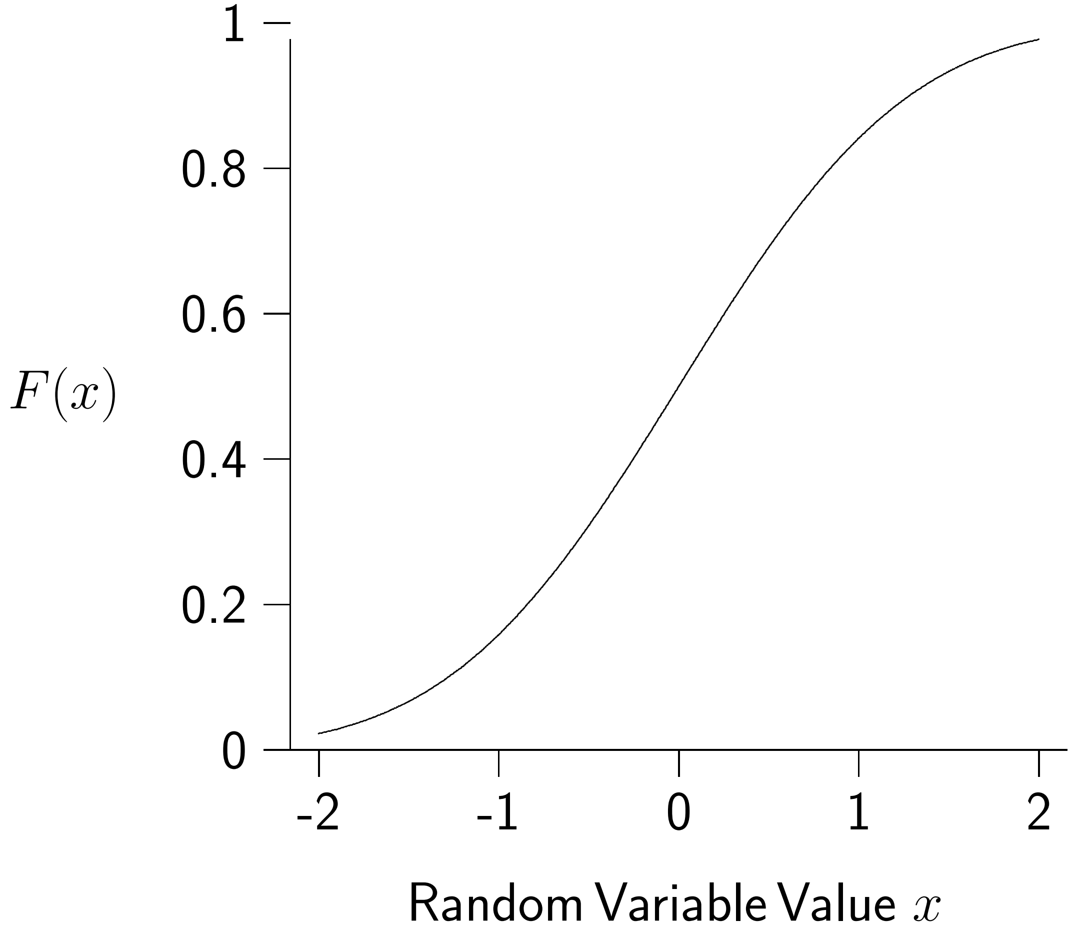


Fig. 20.

iii. Use $\phi_p = \inf\{x | F_X(x) \geq p\}$.

iv. Discrete cases might not have a solution to $p = F_X(\phi)$. See Fig. 21.

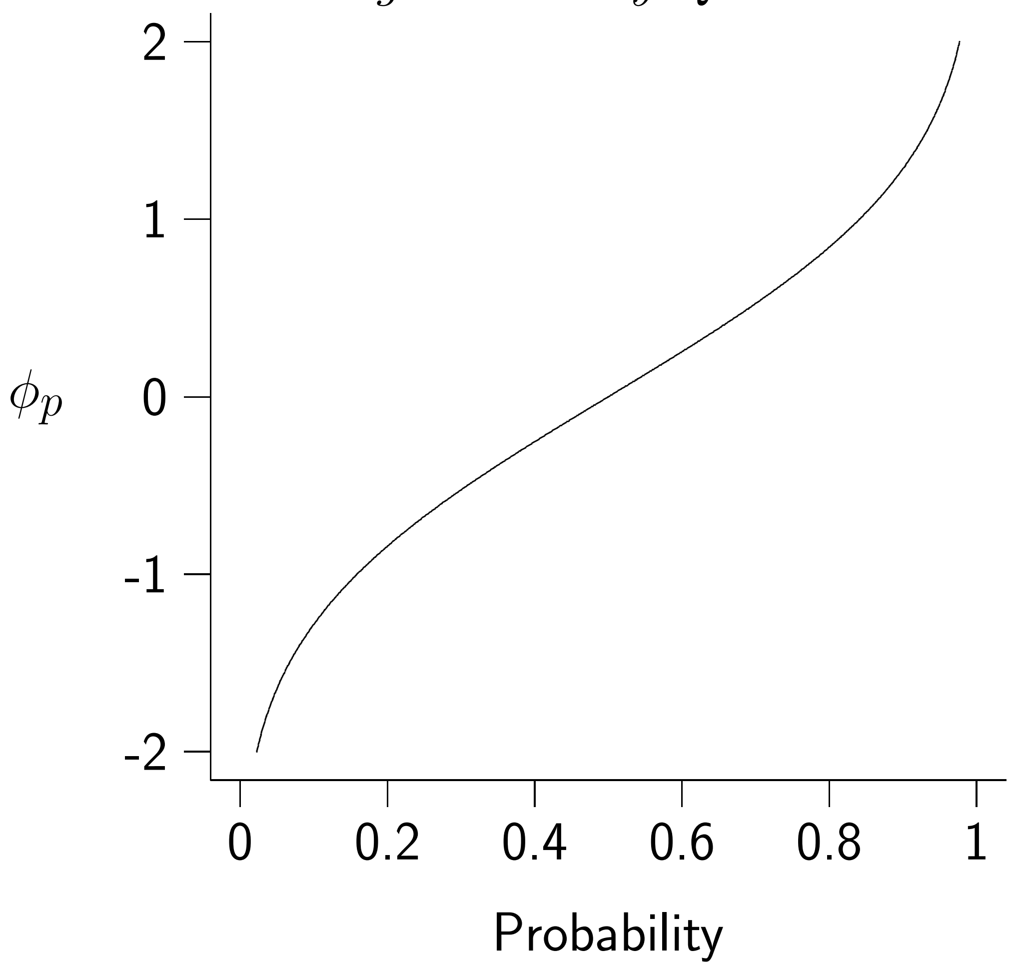
e. Since inverse need not exist, ϕ_p chosen so that

$\mathsf{P}\left(X \leq \phi_p\right) \geq p$, $\mathsf{P}\left(X \geq \phi_p\right) \geq (1-p)$.

2. Some important quantiles

a. The $median$ is the $1/2$ quantile.

Fig. 18: Easy Quantile Function



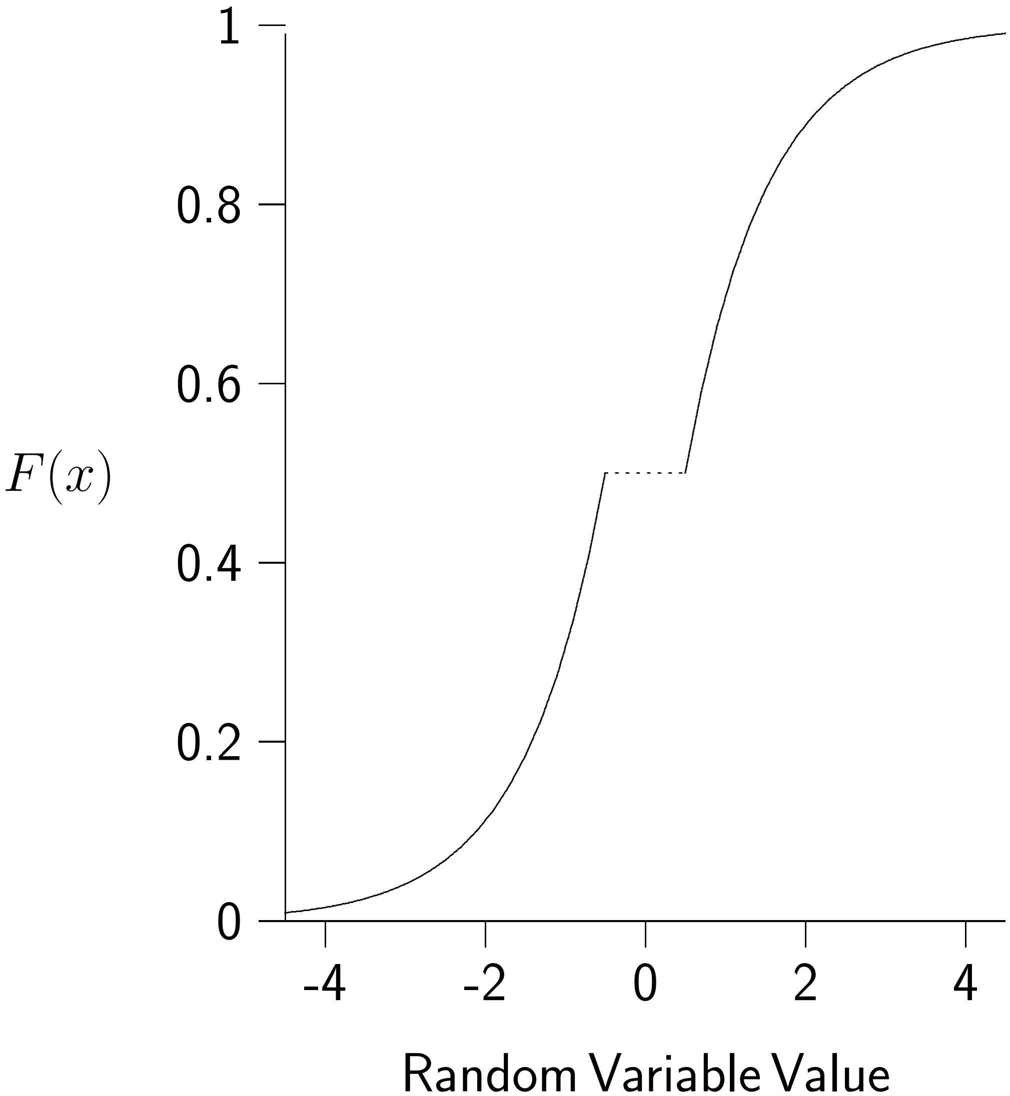
i. Measures the center of a distribution.

ii. The median corresponding to a random variable X with distribution function F_X is that value ν_X such that $\mathsf{P}\left(X \le \nu_X\right) \ge .5$ and $\mathsf{P}\left(X \ge \nu_X\right) \ge .5$.

iii. In terms of the distribution function, the median ν_X satisfies $F_X(\nu_X) \ge .5$, $1 - F_X(\nu_X^-) \ge .5$.

iv. for a discrete distń with probability function p_X it satisfies $\sum_{x \le \nu_X} p_X(x) \ge .5$ and $\sum_{x \ge \nu_X} p_X(x) \ge .5$,

Fig. 19: Continuous Distribution with Distribution Function Plateau



v. for a continuous distń with probability density function f_X it satisfies $\int_{-\infty}^{\nu_X} f_X(x)\,dx = .5$.

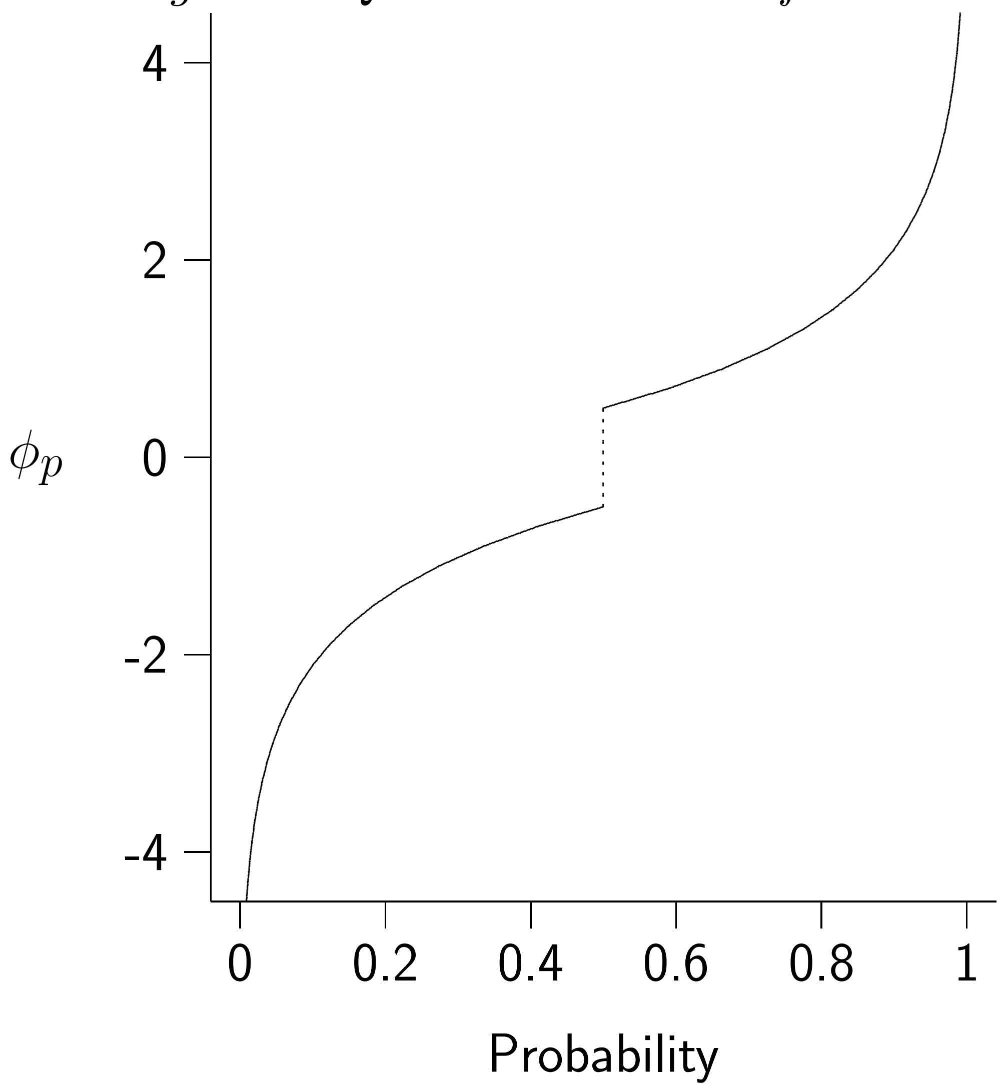
b. A *quartile* is a .25 or .75 quantile.

i. Distinguished by calling upper or lower.

3. Examples of quantiles:

a. Binomial variables with the probability function

Fig. 20: Quantile Function from CDF with Plateu



$$p(x;\pi,1) = \binom{1}{1}\pi^x(1-\pi)^{1-x} = \begin{cases} \pi & \text{if } x = 1 \\ 1-\pi & \text{otherwise} \end{cases}$$

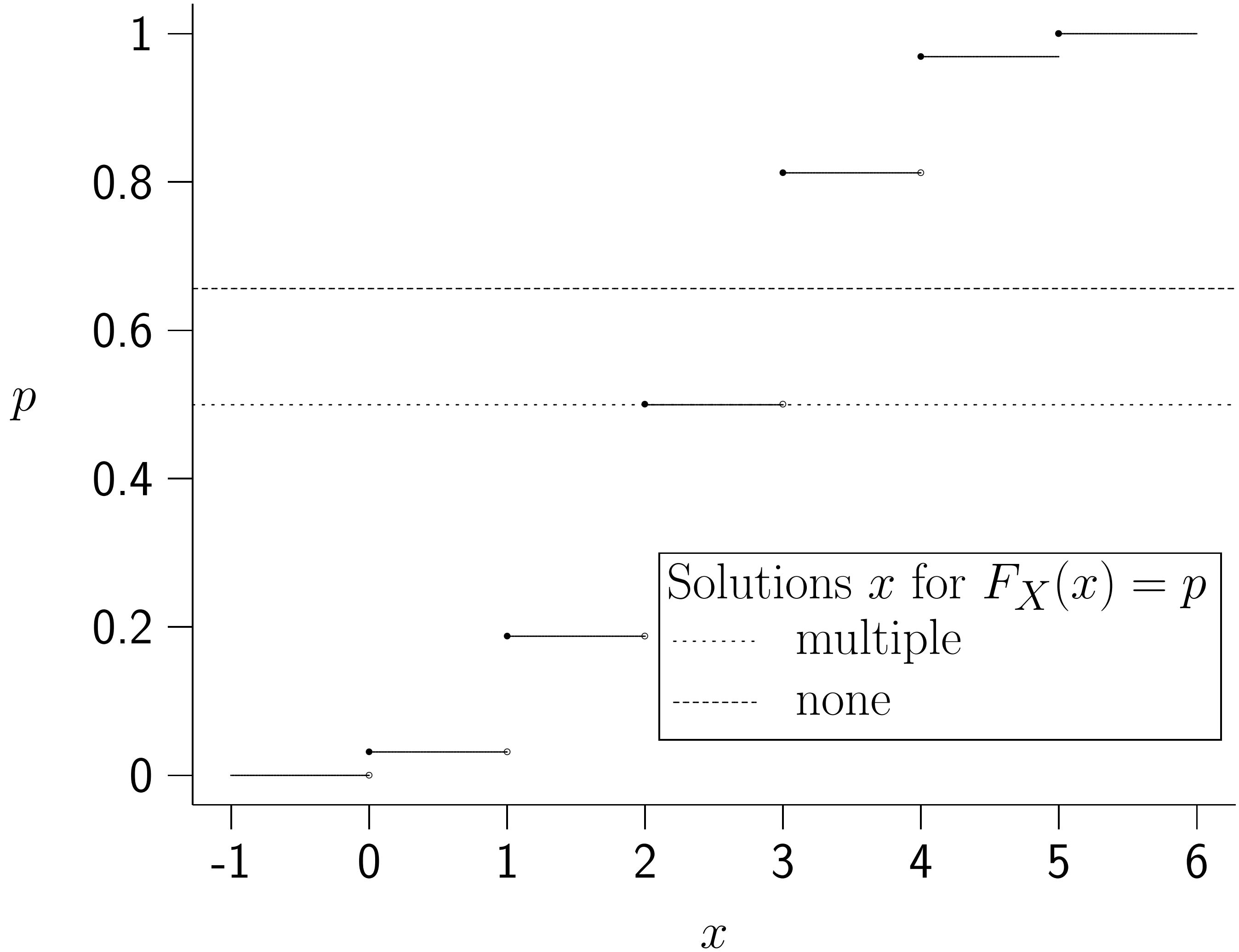
b. Then the median is

$$\begin{cases} 0 & \text{if } \pi < .5 \\ 1 & \text{if } \pi > .5 \\ \text{anything} & \text{otherwise.} \end{cases}$$

i. Binomial variable, probability function $p(x;.5,5)$: See Fig. 21.

4. Comparison of median and expectation

Fig. 14: Quantiles of Discrete Distribution



a. Disadvantages relative to expectation:

   i. The median can't be given explicitly, but only as the solution to an equation involving bounds on integrals or sums,

   ii. sometimes isn't unique,

   iii. sometimes doesn't give much information.

b. Advantage:

   i. always exists.

WMS: 6.3a

C. Transformations of random variables.

1. Univariate Transformations via the Definition

a. X takes values in some set $\mathcal{X}$

b. r is a function defined on $\mathcal{X}$

c. Want to describe distribution of $Y = r(X)$.

i. Y takes values in $\mathcal{Y} = r(\mathcal{X})$.

ii. via the probability function or probability density function for Y.

iii. When X is discrete, this was easy.

- Sum over appropriate values of X:

$$p_Y(y) = \mathsf{P}\left(Y = y\right) = \mathsf{P}\left(r(X) = y\right)$$
$$= \sum_{r(x)=y} \mathsf{P}\left(X = x\right) = \sum_{r(x)=y} p_X(x).$$

iv. If r is one to one,

- an inverse for r exists on $\mathcal{Y}$,
- sums above all have only one addend.

v. If r is non-decreasing, then expression in terms of the distribution function is easy:

$$F_Y(y) = \mathsf{P}\left(Y \leq y\right) = \mathsf{P}\left(r(X) \leq y\right)$$

$$= \mathsf{P}\left(X \le r^{-1}(y)\right) = F_X(r^{-1}(y)). \quad (1)$$

“NA

d. Example:

i. $r(x) = x^2$, distribution function for X is $1 - \exp(-x)$ for $x \ge 0$.

ii. $r^{-1}(y) = \sqrt{y}$.

iii. Then $\mathsf{P}(Y \le y) = \mathsf{P}(X \le \sqrt{x})$

WMS: 6.4a

2. probability density function of a transformed continuous variable

a. When X is continuous then generally (but not always) Y also has a probability density function.

i. Let f_X be the probability density function for X.

ii. Denote the transformation function by $r(x)$.

b. $f_Y(y)$ can generally be expressed in terms of the original probability density function: $f_Y(y) = f_X(r^{-1}(y)) \left| \frac{d}{dy} r^{-1}(y) \right|$.

i. When r is non-decreasing and has inverse,

- Differentiate distribution function:

$$f_Y(y) = \frac{d}{dy} F_X(r^{-1}(y))$$

$$= F'_X(r^{-1}(y))\frac{d}{dy}r^{-1}(y)$$

$$= f_X(r^{-1}(y))\frac{d}{dy}r^{-1}(y) \qquad (2)$$

“NA

ii. Requires transformation to have positive derivative.

c. Examples:

i. $r(x) = x^2$, probability density function for X is $\exp(-x)$ for $x \geq 0$.

- $r^{-1}(y) = \sqrt{y}$, $\frac{d}{dy}r^{-1}(y) = 1/(2\sqrt{y})$.
- $f_Y(y) = \exp(-\sqrt{y})\frac{1}{2}y^{-2}$. See Fig. 22.

ii. $r(x) = \sqrt{x}$, probability density function for X is $c\exp(-x^2)$ for $x \geq 0$.

- $c = 2/\sqrt{\pi}$, but we don't need this.
- $r^{-1}(y) = y^2$, $\frac{d}{dy}r^{-1}(y) = 2y$.
- $f_Y(y) = 2c\exp(-y^4)y$. See Fig. 23.

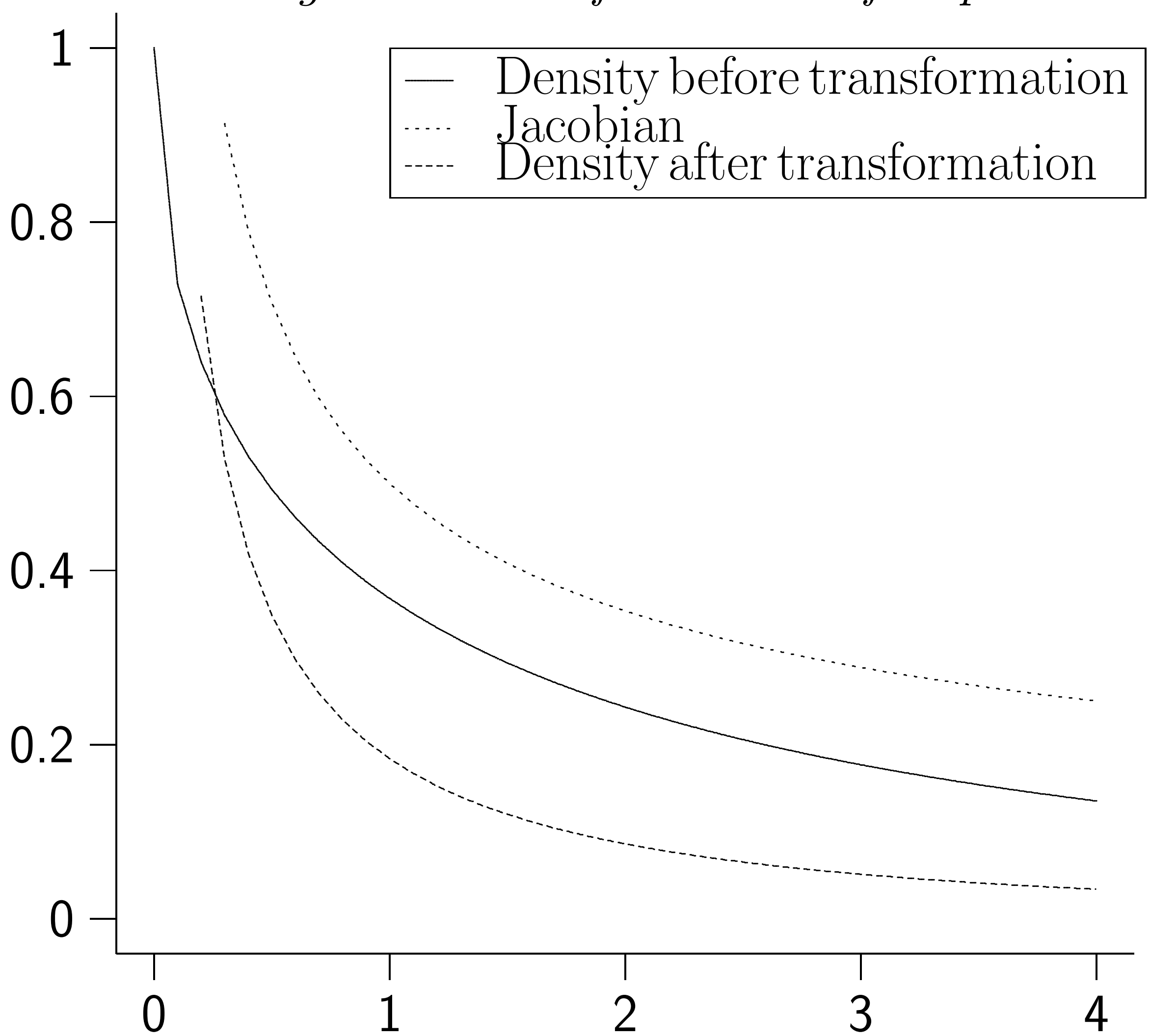
d. Can express using the integration change of variables formula:

i. $\int_A f_X\,dx = \int_B f_X\frac{dx}{dy}dy$;

ii. as functions of y using r then the product $f_X\frac{dx}{dy}$ satisfies requirements for a probability density function.

e. Interpretation: probability density function of Y at y has two

Fig. 22: Transformation of Exponential



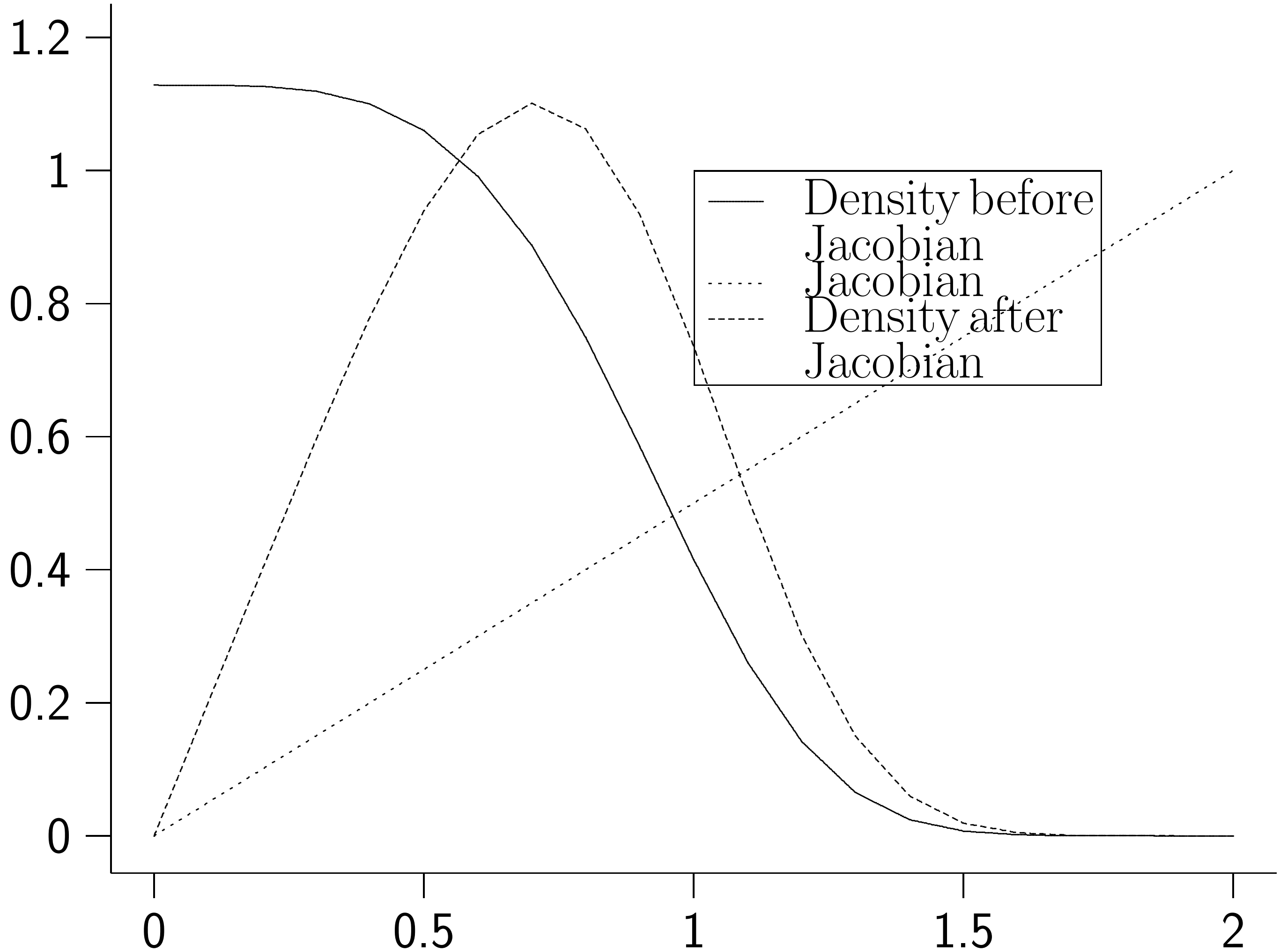
factors:

i. part it inherits form the distribution for X ,

ii. part that arises because of stretching or contracting the scale.

- If r is moving very quickly as x moves, then probability arising from f_X is stretched over a wide range,
- probability density function of Y should be lower than if r were moving more slowly.

Fig. 23: Transformation of Normal



f. Absolute values around derivative account for the case if r non-increasing instead of non-decreasing:

i. By the definition of the distribution function,

$$F_Y(y) = \mathsf{P}\left(Y \leq y\right) = \mathsf{P}\left(r(X) \leq y\right)$$
$$= \mathsf{P}\left(X \geq r^{-1}(y)\right) = 1 - F_X(r^{-1}(y)). \tag{3}$$

ii. probability density function becomes

$$f_Y(y) = \frac{d}{dy}(1 - F_X(r^{-1}(y)))$$

$$= -F'_X(r^{-1}(y))\frac{d}{dy}r^{-1}(y)$$

$$= f_X(r^{-1}(y))(-\frac{d}{dy}r^{-1}(y))$$

$$= f_X(r^{-1}(y))|\frac{d}{dy}r^{-1}(y)| \quad (4)$$

“NA

iii. if r is non-increasing, $\frac{d}{dy}r^{-1}(y) \leq 0$, and so in both non-increasing and non-decreasing cases, probability density function is

$$f_Y(y) = f_X(r^{-1}(y))\left|\frac{d}{dy}r^{-1}(y)\right|$$

.