4. Moment Generating Function Summary

   a. General discrete distribution for X with probabilities π_i on points ζ_i: $m_X(t) = \sum_i \pi_i \exp(\zeta_i t)$ for $t \in \Re$.

   b. Binomial $X \sim$ Bin(π, m): $m_X(t) = (\exp(t)\pi + 1 - \pi)^m$ for $t \in \Re$.

      i. Bernoulli trial as a special case with $m = 1$.

   c. Poisson $X \sim$ Pois(λ): $m_X(t) = \exp([\exp(t) - 1]\lambda)$ for $t \in \Re$.

   d. Negative Binomial $N \sim$ NBin(π, k): $m_N(t) = \exp(kt)[1 - \exp(t)(1 - \pi)]^{-k}\pi^k$ if $t < -\ln(1 - \pi)$.

      i. Geometric as a special case with $k = 1$.

   e. Gamma distribution $X \sim \Gamma(k, \beta)$: $m_X(t) = (1 - \beta t)^{-k}$ if $t < 1/\beta$.

      i. Exponential distribution as a special case with $k = 0$.

   f. Laplace $X \sim$ Lap: $m_X(t) = \frac{(1+t)^{-1}}{2} + \frac{(1-t)^{-1}}{2}$ for $|t| < 1$.

   g. Normal $Y \sim$ N(μ, σ^2): $m_Y(t) = \exp(\frac{1}{2}t^2\sigma^2 + \mu t)$ for $t \in \Re$.

      i. Standard normal as a special case with $\mu = 0$, $\sigma^2 = 1$.

$\vdots$

5. Higher-Dimensional Distributions

a. Sample space consists of a region in $\Re^k$.

i. Example: Continuous biological or economic measurement.

ii. Represent probabilities by integrals of a function over event sets

iii. Get a legitimate probability if function is non-negative and integrates to 1 over all of S.

b. distribution function: $F_{\boldsymbol{X}}(x_1, \cdots, x_k) = \mathsf{P}(X_1 \leq x_1, \ldots, X_k \leq x_k)$

c. probability function: $p_{\boldsymbol{X}}(x_1, \cdots, x_k) = \mathsf{P}(X_1 = x_1, \ldots, X_k = x_k)$

d. probability density function: $f_{\boldsymbol{X}}(x_1, \cdots, x_k)$ such that

$$\int_A f_{\boldsymbol{X}}(x_1, \cdots, x_k)\, dx_1 \cdots dx_k = \mathsf{P}((X_1, \ldots, X_k) \in A)$$

e. probability density function is derivative of distribution function:

i. If $\dfrac{\partial^k F_{X_1,\ldots,X_k}(x_1, \cdots, x_k)}{(\partial x_1 \cdots \partial x_k)}$ exists, then $\partial^k F_{X_1,\ldots,X_k}(x_1, \cdots, x_k)/(\partial x_1 \cdots \partial x_k) = f_{X_1,\ldots,X_k}(x_1, \cdots, x_k)$

6. Marginal and Conditional Quantities

a. Marginal distribution function: If $m < k$,

$$F_{X_1,\ldots,X_m}(x_1,\cdots,x_m)$$
$$= \lim_{x_{m+1}\to\infty}\cdots\lim_{x_k\to\infty} \mathsf{P}\left(X_1 \le x_1,\ldots,X_k \le x_k\right)$$
$$= \lim_{x_{m+1}\to\infty}\cdots\lim_{x_k\to\infty} F_{X_1,\ldots,X_k}(x_1,\cdots,x_k)$$

b. Marginal probability density function: If $m < k$,

$$f_{X_1,\ldots,X_m}(x_1,\cdots,x_m) =$$
$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1,\ldots,X_k}(x_1,\cdots,x_k)\,dx_{m+1}\cdots dx_k$$

c. Conditional distributions: If $\boldsymbol{X}$ continuous, then

$$f_{X_1,\ldots,X_m|X_{m+1},\ldots,X_k}(x_1,\ldots,x_m|x_{m+1},\ldots,x_k)$$
$$= f_{X_1,\ldots,X_k}(x_1,\ldots,x_k)/f_{X_{m+1},\ldots,X_k}(x_{m+1},\ldots,x_k)$$

WMS: 6.3b,6.4b,6.6

D. Transformations of multiple random variables

1. The setup: $U = u(X,Y)$, $V = v(X,Y)$.

a. Inverses $X = x(U,V)$, $Y = y(U,V)$;

i. Unless stated otherwise, assume 1-1.

b. The objective:

i. joint distribution of U, V

ii. marginal distribution of U.

2. Transformations in the Discrete case:

a. $f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v))$

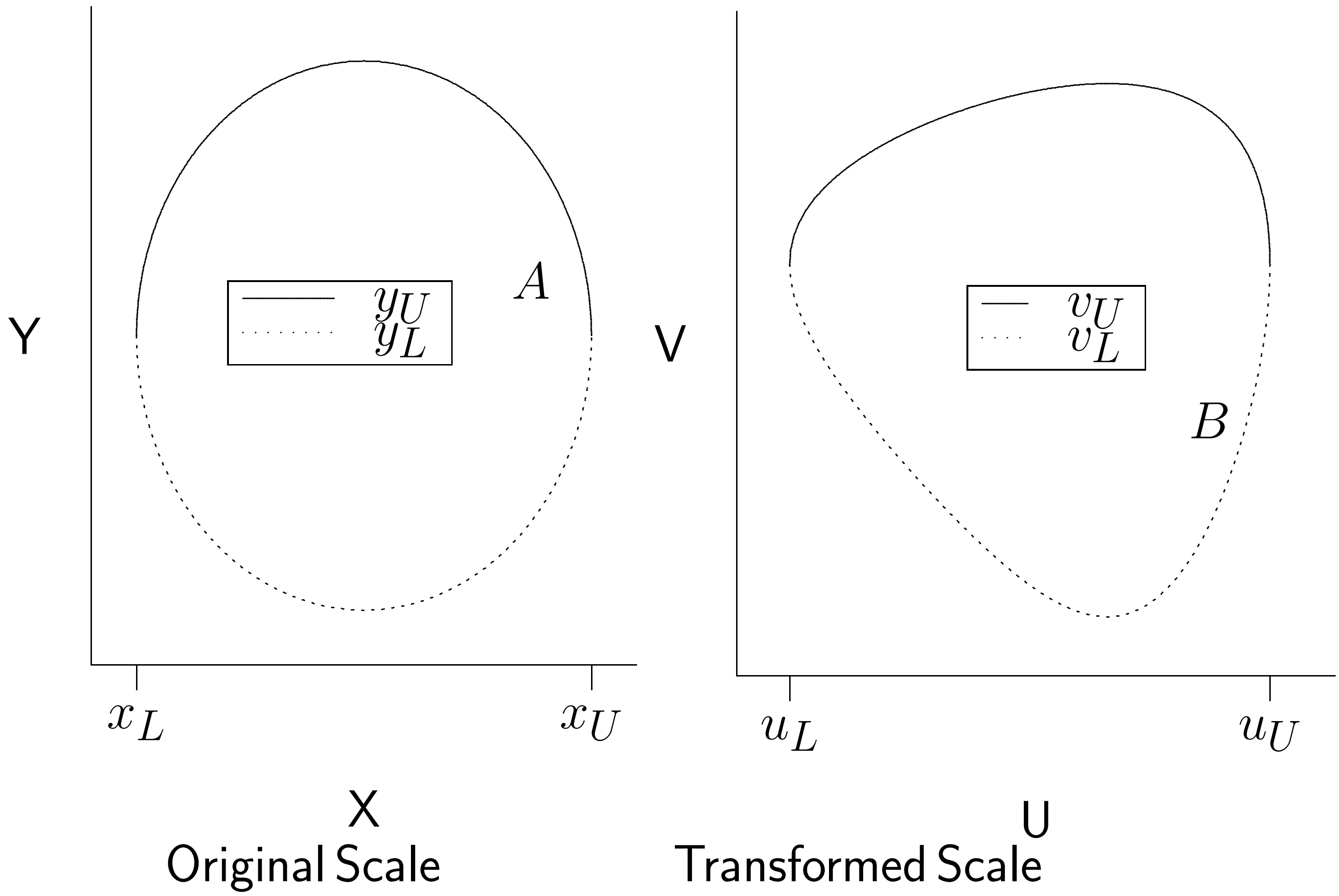
3. Transformations in the Continuous case:

 a. The setup:

  i. Let A be the event on X, Y scale.

  ii. Let B be the event on U, V scale. See Fig. 34.

Fig. 34: Bivariate Transformation



 b. $\mathsf{P}((U,V) \in B) = \mathsf{P}((X,Y) \in A) = \int_{x_L}^{x_U} \int_{y_L(x)}^{y_U(x)} f_{X,Y}(x,y)\, dy\, dx$

c. Suppose u, v differentiable with continuous partial derivatives.

d. Let J be the Jacobian of the forward transformation.

i. $J = \left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right|$

e. Let J^- be the Jacobian of the inverse transformation.

i. $J^- = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = 1/J(x(u,v), y(u,v))$

f. Change variables to U, V: $\mathsf{P}\left((\boldsymbol{U}, \boldsymbol{V}) \in B\right) = \int_{u_L}^{u_U} \int_{v_L(x)}^{v_U(x)} f_{X,Y}(x(u,v), y(u,v)) J^- \, du \, dv$

g. Hence $f_{U,V}(u,v) = f_{X,Y}(x(u,v), y(u,v)) \left|J^-\right|$

h. Multiple inverses: Sum over them.

4. Distribution of a lower-dimensional transformation

a. Ex., (X, Y) have a joint distribution.

b. Want distribution of $U = u(X, Y)$

c. Change of variables formula in multiple dimensions doesn't work,

i. because Jacobian isn't defined,

ii. and so density formula doesn't work.

d. Fix via transforming to a space of the same dimension and marginalizing.

i. Choosing a convenient additional variable V

ii. A function of X and Y

iii. Remove its effect on the density by marginalizing.

WMS: 6.5

5. The mgf method: $U = u(X, Y)$

a. $m_U(v) = \mathsf{E}(\exp(vu(X, Y)))$

b. Try to recognize this.

WMS: 5.5

E. Expectations in Multiple Dimensions

1. Setup:

a. Consider jointly-distributed random variables (X, Y)

b. Consider some summary function $g_1(x, y)$.

c. Want $\mathsf{E}(g_1(X, Y))$.

2. Discrete distributions:

a. Let $Z = g_1(X, Y)$.

b. Calculate probability function $p_Z(z) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y} | z = g_1(x,y)} p_{X,Y}(x, y)$

c. Determine sample space $\mathcal{Z}$.

d. Sum $\sum_{z \in \mathcal{Z}} z p_Z(z)$

e. Equivalent to

$$\sum_{x\in\mathcal{X},y\in\mathcal{Y}} g_1(x,y)p_{X,Y}(x,y).$$

3. Continuous case:

   a. Calculate probability density function $f_Z(z)$

      i. Make a second new random variable $W = g_2(X,Y)$ so that $J \neq 0$.

      ii. Construct sample space $\mathcal{Z}$, and, for each $z \in \mathcal{Z}$, construct the sample space for W $\mathcal{W}_z$.

      iii. Calculate inverse functions h_1, h_2 to g_1, g_2

      iv. Joint probability density function is $f_{Z,W}(z,w) = f_{X,Y}(h_1(z,w), h_2(z,w))J^-$

      v. Marginal probability density function is $\int f_{Z,W}(z,w)\,dw = \int f_{X,Y}(h_1(z,w), h_2(z,w))J^-\,dw$

   b. Expectation is $\int_{\mathcal{Z}} z f_Z(z)dz$

      i. Substitute in: $\int_{\mathcal{Z}} z \int_{\mathcal{W}_z} f_{Z,W}(z,w)\,dw\,dz = \int_{\mathcal{Z}}\int_{\mathcal{W}_z} z f_{X,Y}(h_1(z,w), h_2(z,w))J^-\,dw\,dz$

      ii. Change variables back

c. Result is

$$\mathsf{E}(Z) = \int_{\mathcal{X}} \int_{\mathcal{Y}} g_1(x,y) f_{X,Y}(x,y)\, dy\, dx.$$

i. Did not depend on our choice of g_2.

WMS: 5.6

4. Summability:

a. Suppose X and Y have a joint probability function $f_{X,Y}(x,y)$.

b. Then $\mathsf{E}(X+Y) = \mathsf{E}(X) + \mathsf{E}(Y)$

i. Because

$$\begin{aligned}
\mathsf{E}(X+Y) &= \iint (x+y) f_{X,Y}(x,y)\, dx\, dy \\
&= \iint x f_{X,Y}(x,y)\, dx\, dy + \iint y f_{X,Y}(x,y)\, dx\, dy \\
&= \iint x f_{X,Y}(x,y)\, dy\, dx + \iint y f_{X,Y}(x,y)\, dx\, dy \\
&= \int x f_X(x)\, dx + \int y f_Y(y)\, dy \\
&= \mathsf{E}(X) + \mathsf{E}(Y)
\end{aligned}$$

ii. Holds for discrete variables by substituting summation for integral.

c. By extension this holds for any number of summands.

d. Ex.: a Bin(m, π) variable has expectation $m\pi$.

e. Hence the expectation has the advantage of transforming easily.

5. Multiplicative property under independence:

a. If X and Y are independent,

b. g and h are functions

c. Then $\mathsf{E}\left(g(X)h(Y)\right) = \mathsf{E}\left(g(X)\right)\mathsf{E}\left(h(Y)\right)$.

i. Because

$$
\begin{aligned}
\mathsf{E}\left(g(X)h(Y)\right) &= \int\int g(x)h(y)f_{X,Y}(x,y)\,dx\,dy \\
&= \int\int g(x)h(y)f_X(x)f_Y(y)\,dx\,dy \\
&= \int g(x)f_X(x)\,dx \int h(y)f_Y(y)\,dy \\
&= \mathsf{E}\left(g(X)\right)\mathsf{E}\left(h(Y)\right)
\end{aligned}
$$

ii. Holds for discrete variables by substituting summation for integral.