960:583– Methods of Inference– Spring, 2019

Exam 2

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Total	

1. Suppose that $X_1, \cdots, X_n$ are independent normal random variables with expectation μ and variance 1 , with $n = 10$. Calculate the power for the level 0.05 test of the null hypothesis $\mu = 0$ vs. the alternative hypothesis $\mu = 1$.

(30 pts) The standard test (which is the Neyman-Pearson test) rejects the null hypothesis if $\bar{X} \geq 1.64/\sqrt{n}$. *Power is* $\mathrm{P}_A\left[\bar{X} \geq 1.64/\sqrt{n}\right] = \mathrm{P}_A\left[\bar{X} - 1 \geq 1.64/\sqrt{n} - 1\right] = \mathrm{P}_A\left[\sqrt{n}(\bar{X} - 1) \geq 1.64 - 1\sqrt{n}\right] = \bar{\Phi}(1.64 - \sqrt{n}) = 0.936$.

Total for this question: 30.

2. Consider random variables $X_1, \cdots, X_n$, independent and with mass function $\lambda^x(1-\lambda)$, for $\lambda \in (0,1)$. Place a prior distribution with density proportional to $\lambda^{-1}(1-\lambda)^{-1}$.

a. Note that $\int_0^1 \lambda^{-1}(1-\lambda)^{-1}\,d\lambda = \infty$. What term describes this quality of a prior?

(5 pts) Improper. Noninformative was also accepted. THE PRIOR IS ALSO CONJUGATE, BUT THE PROPERTY I ASKED ABOUT DOESN'T HAVE ANYTHING TO DO WITH CONJUGACY.

b. Calculate the posterior density for λ , conditional on $X_1, \cdots, X_n$.

(20 pts)

$$\begin{aligned} f(\lambda) &= \lambda^{-1}(1-\lambda)^{-1}\prod_{i=1}^n \lambda^{X_i}(1-\lambda) \Big/ \int_0^1 \lambda^{-1}(1-\lambda)^{-1}\prod_{i=1}^n \lambda^{X_i}(1-\lambda)\,d\lambda \\ &= (1-\lambda)^{n-1}\lambda^{\sum_{i=1}^n X_i - 1} \Big/ \int_0^1 (1-\lambda)^{n-1}\lambda^{\sum_{i=1}^n X_i - 1}\,d\lambda \\ &= (1-\lambda)^{n-1}\lambda^{\sum_{i=1}^n X_i - 1}/B(n, \sum_{i=1}^n X_i). \end{aligned}$$

c. Write down two equations giving endpoints of the highest posterior density region of probability α . Do not try to solve them.

(15 pts) $(1-L)^{n-1}L^{\sum_{i=1}^n X_i - 1} = (1-U)^{n-1}U^{\sum_{i=1}^n X_i - 1}$ *and* $\int_L^U (1-\lambda)^{n-1}\lambda^{\sum_{i=1}^n X_i - 1}\,d\lambda = B(n, \sum_{i=1}^n X_i)(1-\alpha)$. ANY INTERVAL SATISFYING THE SECOND EQUATION ABOVE IS CREDIBLE, BUT YOU NEED THE FIRST AS WELL TO MAKE THE INTERVAL HPD. FOR AN ASYMETRIC POSTERIOR, THE EQUAL-TAILED CREDIBLE REGION WILL GENERALLY NOT BE HPD.

d. Calculate the posterior mode of λ , as an estimator of λ .

(15 pts) The log density is $(n-1)\log(1-\lambda)+(\sum_{i=1}^n X_i - 1)\log(\lambda)$ *. Differentiating gives* $-\frac{n-1}{1-\lambda}+\frac{\sum_{i=1}^n X_i-1}{\lambda}$ *. Substituting the mode and and setting to zero gives* $\frac{n-1}{1-\tilde\lambda}=\frac{\sum_{i=1}^n X_i-1}{\tilde\lambda}$ *. Solving gives* $\tilde\lambda=\frac{\sum_{i=1}^n X_i-1}{n+\sum_{i=1}^n X_i-2}$ *. The second derivative of the log density is* $-\frac{n-1}{(1-\lambda)^2}-\frac{\sum_{i=1}^n X_i-1}{\lambda^2}<0$ *, implying that the critical value is a maximizer.*

Total for this question: 55.

3. Suppose that X has a Poisson distribution with mass function

$$p_X(x;\theta)=\exp(-\theta)\theta^x/x!,$$

for $\theta>0$ and $x\in\{0,1,2,3,\ldots\}$. Suppose Y has a Poisson distribution with mass function

$$p_Y(y;\rho)=\exp(-\rho)\rho^y/y!,$$

for $\rho>0$ and $y\in\{0,1,2,3,\ldots\}$. Suppose X and Y are independent.

a. Construct the (generalized) likelihood ratio test statistic Λ testing the null hypothesis that $\theta=\rho$ vs. the alternative hypothesis that $\theta\neq\rho$.

(35 pts) The likelihood is

$$L(\theta,\rho)=\exp(-\theta)\theta^X/X!\exp(-\rho)\rho^Y/Y!.$$

Maximizing the log over the union of the null and alternative, $\hat\theta$ *satisfies* $\frac{d}{d\theta}(-\theta+X\log(\theta)-\log(X!))=0$ *, or* $-1+X/\theta=0$ *, or* $\hat\theta=X$ *. Similarly, the maximizer for* ρ *is* $\hat\rho=Y$ *. The likelihood under the null hypothesis is*

$$L(\upsilon,\upsilon)=\exp(-\upsilon)\upsilon^X/X!\exp(-\upsilon)\upsilon^Y/Y!,$$

with the maximizer satisfying $-2+(X+Y)/\hat\upsilon=0$ *, or* $\hat\upsilon=(X+Y)/2$ *. Hence the likelihood ratio statistic is*

$$\Lambda=\frac{\exp(-X-Y)((X+Y)/2)^{X+Y}}{\exp(-X)X^X\exp(-Y)Y^Y}=\frac{((X+Y)/2)^{X+Y}}{X^XY^Y}.$$

b. Suppose a value of Λ of $1/4$ is observed. Use an approximate test of level 0.05 to determine whether the null hypothesis is rejected.

(15 pts) Use Wilks' lemma to note that under the null hypothesis, $-2\ln(\Lambda)\ \chi^2_1$ *. Hence reject if* $-2\ln(\Lambda)>3.84$ *. In our case,* $-2\ln(\Lambda)=2\ln(4)=2.773$ *. Do not reject the null hypothesis.* This test was demonstrated in class under the assumption of large sample size, leading to asymptotic normality of the score function. We do not have this here; note, however, that the sum of independent Poisson random variables is Poission, with a rate parameter equal to the sum of the rate parameters of the individual observations, and furthermore, the sum of independent Poisson variables is sufficient for inference on a common rate. Hence the random variables X and Y in the statement of the question can be considered as the sum of a large number of Poisson summands with a correspondingly smaller rate. Then the large sample assumption of Wilk's lemma is equivalent to a large rate parameter for the Poisson totals. You might note this, but it wasn't required.

c. Suppose a value of Λ of $1/4$ is observed. Approximate the p-value.
(10 pts) The $\mathrm{P}_0[\Lambda < 1/4] = \mathrm{P}[-2\ln(\Lambda) > 2.773] = .1$. *The p-value is .1.*

Total for this question: 60.

4. Consider random variables $X_1, \cdots, X_n$, independent and with mass function $\lambda^x(1-\lambda)$ for $x \in \{0,1,2,\ldots\}$ and $\lambda \in [0,1]$. Construct tests of level α. You do NOT need to calculate a critical value.

a. Construct a most powerful test for testing the null hypothesis that $\lambda = 1/4$, vs. the alternative that $\lambda = 3/4$, or tell why this is impossible.
(25 pts) The likelihood is $L(\lambda) = \prod_{i=1}^n \lambda^{X_i}(1-\lambda) = (1-\lambda)^n \lambda^{\sum_{i=1}^n X_i}$. *The likelihood ratio statistic is*

$$\Lambda = \frac{(3/4)^n (1/4)^{\sum_{i=1}^n X_i}}{(1/4)^n (3/4)^{\sum_{i=1}^n X_i}} = 3^{n-\sum_{i=1}^n X_i}.$$

Reject when Λ small. Equivalently, reject when $\sum_{i=1}^n X_i > c$, for some critical value c. If you do not simplify this to a rejection region defined in terms of $\bar{X}$, the next two parts of the question will be harder to think through.

b. Construct a uniformly most powerful test for testing the null hypothesis that $\lambda = 1/4$, vs. the alternative that $\lambda > 1/4$, or tell why this is impossible.
(15 pts) In this case, the Neyman Pearson statistic is

$$\Lambda = \frac{(3/4)^n (1/4)^{\sum_{i=1}^n X_i}}{(1-\lambda)^n \lambda^{\sum_{i=1}^n X_i}} = (3/(4(1-\lambda)))^n (4\lambda)^{-\sum_{i=1}^n X_i}.$$

Since $4\lambda > 1$, reject when $\sum_{i=1}^n X_i$ large. The critical value depends only on the null hypothesis, and so the most powerful test does not depend on which member of the alternative hypothesis we consider. The point of this question is to review the most powerful quality of tests for a compound alternative when the form of the tests for each separate simgle alternative does not depend on the alternative. Using a generalized likelihood ratio test here takes you off on the wrong track, because generalized likelihood ratio tests generally have no power-maximizing properties.

c. Construct a uniformly most powerful test for testing the null hypothesis that $\lambda = 1/4$, vs. the alternative that $\lambda \neq 1/4$, or tell why this is impossible.
(15 pts) Again

$$\Lambda = (3/(4(1-\lambda)))^n (4\lambda)^{-\sum_{i=1}^n X_i}.$$

In this case, for $\lambda < 1/4$, the most powerful test rejects for $\sum_{i=1}^n X_i$ small. Hence no test can be uniformly most powerful.