WMS: 1

I. Aims of Statistics:
  A. Probability is the study of relative proportions of random outcomes based on structure of generating process.
  B. Statistics is the inverse problem:
    1. Observe data generated by process
    2. Infer process.

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II. Estimation
  A. Aim
    1. Want to estimate some number θ, called a *parameter*.
      a. Fraction of population supporting a candidate
      b. Population Average effect of some cholesterol–lowering medication.
      c. Mass of an electron
    2. Rule that gives estimate is called an estimator.
    3. Want it based on some data.

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  B. Preliminaries: What makes a good estimator
    1. Quantify what happens if you make a wrong decision
      a. Suppose that you pay a penalty $L(a,\theta)$ if your guess is a when the truth is θ
        i. Penalty is called *Loss function*.
        ii. Most typically, $L(a,\theta)=(a-\theta)^2$: Squared error loss.
  C. Typically, want rule that depends on data, $\delta(\boldsymbol{X})$
    1. Example
      a. If $X\sim \mathcal{B}\text{in}(n,\theta)$, $\delta(X)$ might be X/n

      b. If $\boldsymbol{X}$ is a sample from a population with expectation θ, then $\delta(\boldsymbol{X})$ might be $\bar{X}$
      c. If $\boldsymbol{X}$ is a sample from a population with median θ, then $\delta(\boldsymbol{X})$ might be sample median
    2. Select estimator before wee see data.
      a. Consider average of loss function $R(\delta,\theta)=\mathrm{E}\left[L(\delta(\boldsymbol{X}),\theta)\right]$.
      b. R is called *risk function*.
        i. Risk function for squared error loss is called *mean squared error (MSE)*.
      c. Review expectation
      d. Let $\mu=\mathrm{E}\left[\delta(\boldsymbol{X})\right]$
      e. $R(\delta,\theta)$ is variance plus bias squared.

$$
\begin{aligned}
&=\mathrm{E}\left[(\delta(\boldsymbol{X})-\mu+\mu-\theta)^2\right]\\
&=\mathrm{E}\left[(\delta(\boldsymbol{X})-\mu)^2+2(\delta(\boldsymbol{X})-\mu)(\mu-\theta)+(\mu-\theta)^2\right]\\
&=\mathrm{E}\left[(\delta(\boldsymbol{X})-\mu)^2\right]+\mathrm{E}\left[2(\delta(\boldsymbol{X})-\mu)(\mu-\theta)\right]\\
&\quad+\mathrm{E}\left[(\mu-\theta)^2\right]\\
&=\mathrm{Var}\left[\delta(\boldsymbol{X})\right]+2(\mu-\theta)\mathrm{E}\left[(\delta(\boldsymbol{X})-\mu)\right]+(\mu-\theta)^2\\
&=\mathrm{Var}\left[\delta(\boldsymbol{X})\right]+(\mathrm{E}\left[\delta(\boldsymbol{X})\right]-\theta)^2
\end{aligned}
$$

      f. We can often make the second part 0:
        i. If $\mathrm{E}\left[\delta(\boldsymbol{X})\right]=\theta$, then $\delta(\boldsymbol{X})$ is called *unbiased*.
        ii. and $\mathrm{E}\left[\delta(\boldsymbol{X})\right]-\theta$ is called the *bias*.
      g. We will see that unbiasedness does not completely specify the best estimator
        i. Ex.: $X\sim\mathcal{N}(\theta,\sigma^2)$ with σ known.
        ii. $\delta(X)=aX+b$

        iii. $R(\delta,\theta)=(a\theta+b-\theta)^2+a^2\sigma^2=(a-1)^2\theta^2+2b(a-1)\theta+b^2+a^2\sigma^2$
        iv. If $a\neq 1$ maximum is $\infty \Rightarrow$ choose $a=1$, and risk is $b^2+a^2\sigma^2 \Rightarrow$ choose $b=0$
      h. “Best” (ie, minimax) estimator might allow some bias in return for smaller variance

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  D. Examples
    1. Estimate of the range of a uniform distribution from the range of a sample
      a. $X_1,\ldots,X_n\sim\mathcal{U}[\alpha,\beta]$, i.i.d..
      b. $X_{(1)},\ldots,X_{(n)}$ are ordered values: order statistics.
      c. $\delta(\boldsymbol{X})=X_{(n)}-X_{(1)}$
      d. Density of $X_{(n)}$ is

$$n\frac{1}{\beta-\alpha}\left(\frac{y-\alpha}{\beta-\alpha}\right)^{n-1}$$

      e. $\mathrm{E}\left[X_{(n)}\right]$ is

$$
\begin{aligned}
&=\int_\alpha^\beta yn\frac{1}{\beta-\alpha}\left(\frac{y-\alpha}{\beta-\alpha}\right)^{n-1}\,dy\\
&=\int_0^1(\alpha+z(\beta-\alpha))nz^{n-1}\,dz\\
&=n\alpha\int_0^1 z^{n-1}\,dz+n(\beta-\alpha)\int_0^1 z^n\,dz\\
&=\alpha+\frac{n}{n+1}(\beta-\alpha)
\end{aligned}
$$

      f. $\mathrm{E}\left[Y_{(1)}\right]=-(-\beta+\frac{n}{n+1}(-\alpha-(-\beta))=\beta-\frac{n}{n+1}(\beta-\alpha)$

      g. $\mathrm{E}\left[\text{Range}\right]=\alpha-\beta+\frac{2n}{n+1}(\beta-\alpha)=e_n(\beta-\alpha)$ for $e_n=\frac{n-1}{n+1}$.
      h. Bias is $\frac{-2}{n+1}(\beta-\alpha)$
        i. Almost unbiased.
          - $\lim_{n\to\infty}$ bias $=0$.
          - Called *asymptotically unbiased*.
      i. Hence to get unbiased estimator, use $\frac{n+1}{n-1}$Range.
        i. Let $v_n=\mathrm{Var}\left[X_{(n)}-X_{(1)}\right](\beta-\alpha)^{-2}$, which does not depend on α or β.
        ii. MSE of $\delta(\boldsymbol{X})=a\text{Range}(\boldsymbol{X})$ is $\{(ae_n-1)^2+a^2v_n\}(\beta-\alpha)^2$
        iii. Differentiating, setting the derivative to zero, and solving for a gives $a=e_n/(e_n^2+v_n)$.
        iv. Since $v_n>0$, minimizing a leaves $\delta(\boldsymbol{X})$ with a slight bias.
    2. Estimating variance
      a. $X_1,\ldots,X_n$ i.i.d., expectation μ, variance σ^2.
      b. $\bar{X}=\sum_{j=1}^n X_j/n$
      c. Then $\sum_{j=1}^n(X_j-\mu)^2$ is

$$=\sum_{j=1}^n(X_j-\bar{X})^2+\sum_{j=1}^n(\bar{X}-\mu)^2+2\sum_{j=1}^n(\bar{X}-\mu)(X_j-\bar{X})^2$$

$$=\sum_{j=1}^n(X_j-\bar{X})^2+n(\bar{X}-\mu)^2.$$

      d. $\sum_{j=1}^n(X_j-\bar{X})^2=\sum_{j=1}^n(X_j-\mu)^2-n(\mu-\bar{X})^2$
      e. $S^2=\sum_{j=1}^n(X_j-\bar{X})^2/(n-1)$

i. $= [\sum_{j=1}^{n}(X_j-\mu)^2 - n(\bar{X}-\mu)^2]/(n-1)$

f. $\mathrm{E}\left[S^2\right] = \dfrac{\sum_{i=1}^{n}\mathrm{E}\left[(X_i-\mu)^2\right]}{n-1} - n\dfrac{\mathrm{E}\left[(\bar{X}-\mu)^2\right]}{n-1}$.

$= n\sigma^2/(n-1) - \sigma^2/(n-1) = \sigma^2$

i. Makes no assumption about distribution of observations.

3. Estimating Standard Deviation
   a. $X_1,\ldots,X_n \sim \mathcal{N}[\mu,\sigma^2]$, i.i.d..
   b. $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$.
      i. χ_k^2 distribution is distribution of sum of squares of k independent $\mathcal{N}(0,1)$ random variables
   c. $\mathrm{E}\left[S\sqrt{n-1}/\sigma\right]$

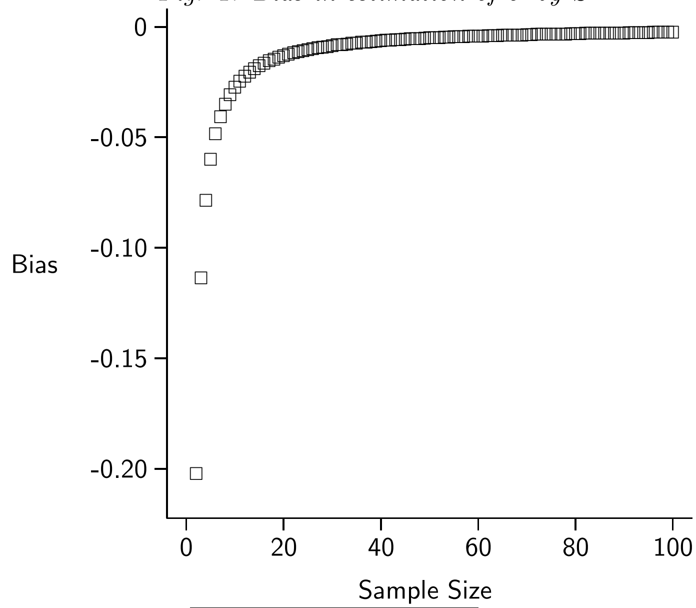
$$= \int_0^\infty \frac{\sqrt{x}\exp(-1/2x)x^{(n-1)/2-1}\,dx}{2^{(n-1)/2}\Gamma((n-1)/2)}$$

$$= \sqrt{2}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\int_0^\infty \frac{\exp(-1/2x)x^{n/2-1}\,dx}{2^{n/2}\Gamma(n/2)}$$

$$= \sqrt{2}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}.$$

   d. $\mathrm{E}[S] = \sigma\dfrac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{n-1}}$
   e. Bias is $\sigma\left(\dfrac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{n-1}} - 1\right)$
   f. See Fig. 1.

Fig. 1: Bias in estimation of σ by S



Calculations for $\sqrt{\sum_{k=1}^{n}(X_k-\bar{X})^2/(n-1)}$ with normal observations

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