Multivariate Tail Probability Approximations

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3. Alternative hypothesis is that canine therapy is superior for at least one of the ordering.
4. p-value for intersection union test is calculated from $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1 \text{ or } \bar{X}^2 \geq \bar{x}^2 | \text{other sufficient statistics}\right]$.

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5. Let $\bar{\boldsymbol{X}} = \sum_{i=1}^n \boldsymbol{X}_i / n$

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2. Use existence of cumulant generating function to obtain relative error behavior uniform for values of $\bar{\boldsymbol{x}}$ in an open ball about the mean.
    1. Measure theoretical error behavior in terms of inverse powers of $\sqrt{n}$.

Inversion Integrals:

1. Density $f_n(\bar{w}^1, \ldots, \bar{w}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{w}^i \tau_i]) \; d\tau_1 \cdots d\tau_p$$

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    1. Write $\bar{\Phi}(\boldsymbol{z}) = \mathrm{P}\left[Z^1 \geq z^1 \cap \ldots \cap Z^p \geq z^p\right]$ for Z_i independent standard normals.

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- Call such an approximation *reflexive*.

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4. Set $c_1 = \hat{\tau}$.
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6. Integrate term-wise.
7. Then $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1\right]$

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      2. This fails for $p > 1$.

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Adjustment induces sample space rotation for $\hat{\Omega}_j$

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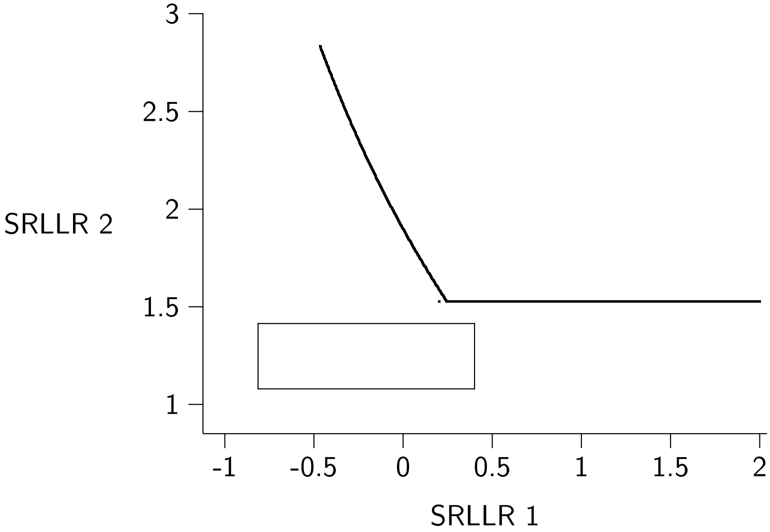

Correlated gamma example, $\boldsymbol{x} = (4, 5)$

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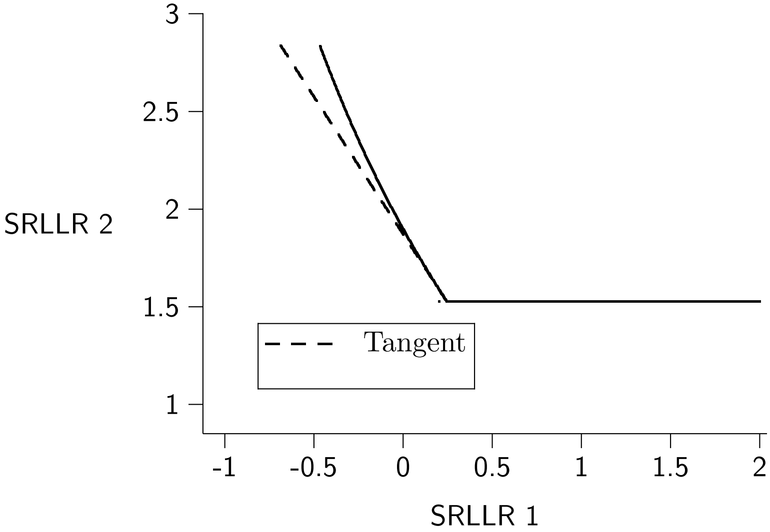

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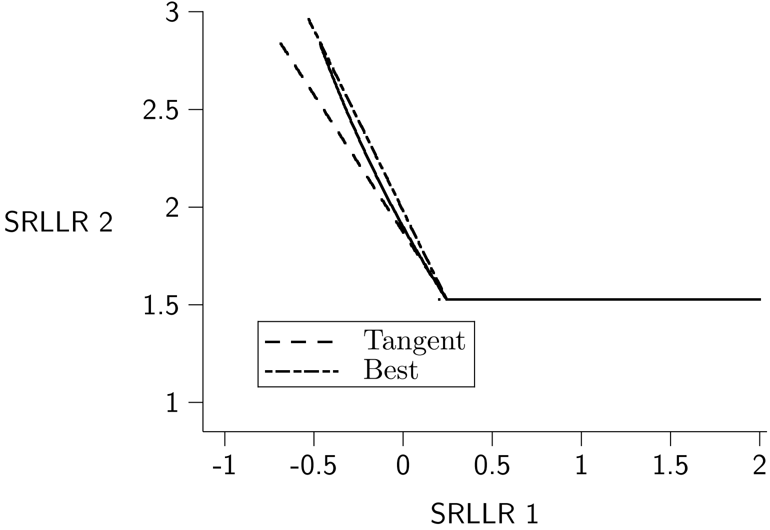

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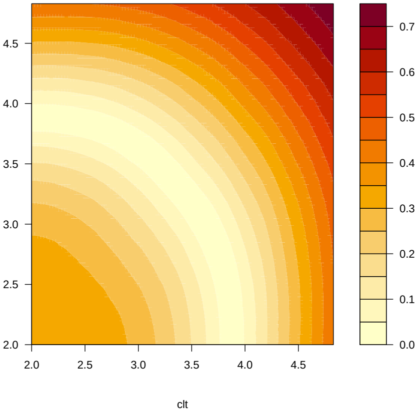
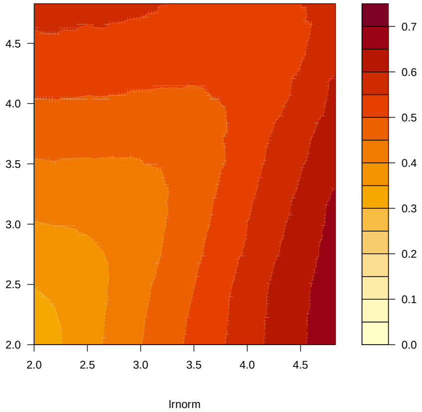
3. $n = 1$!

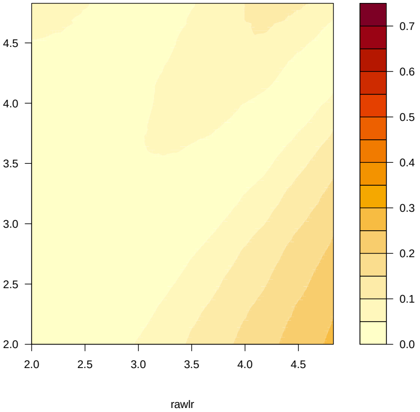
Relative Error: Normal on raw scale



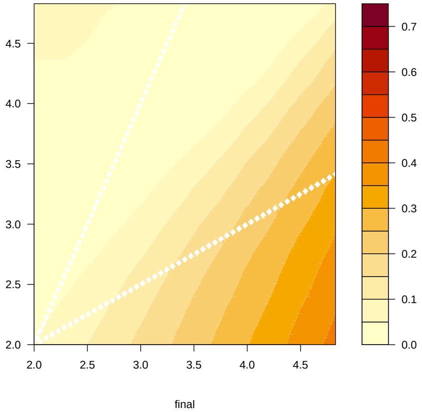
Relative Error: Normal on SRLLR scale



Relative Error: SRLLR scale, adjust for marginal non-normality



Relative Error: SRLLR scale, all adjustments



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5. Extension to conditional distributions.

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