Bivariate Tail Probability Approximations

John E. Kolassa and Donghyun Lee
Rutgers, the State University of New Jersey

21 December 2023

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
    1. Studies US military subject to physical trauma

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
   1. Studies US military subject to physical trauma
   2. Measures efficacy of canine therapy on PTSD, vs standard therapy.

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
    1. Studies US military subject to physical trauma
    2. Measures efficacy of canine therapy on PTSD, vs standard therapy.
    3. Performance was evaluated on both therapies for subject j: $X_j^i = 1$ if canine therapy superior for subject j with ordering $i \in \{1, 2\}$.

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
    1. Studies US military subject to physical trauma
    2. Measures efficacy of canine therapy on PTSD, vs standard therapy.
    3. Performance was evaluated on both therapies for subject j: $X_j^i = 1$ if canine therapy superior for subject j with ordering $i \in \{1, 2\}$.
2. Hypotheses:

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
    1. Studies US military subject to physical trauma
    2. Measures efficacy of canine therapy on PTSD, vs standard therapy.
    3. Performance was evaluated on both therapies for subject j: $X_j^i = 1$ if canine therapy superior for subject j with ordering $i \in \{1, 2\}$.
2. Hypotheses:
    1. Null hypothesis is that there is no systematic difference in therapies, regardless of ordering.

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
   1. Studies US military subject to physical trauma
   2. Measures efficacy of canine therapy on PTSD, vs standard therapy.
   3. Performance was evaluated on both therapies for subject j: $X_j^i = 1$ if canine therapy superior for subject j with ordering $i \in \{1, 2\}$.
2. Hypotheses:
   1. Null hypothesis is that there is no systematic difference in therapies, regardless of ordering.
   2. Alternative hypothesis is that canine therapy is superior for at least one of the ordering.

Motivation Example

1. Hao and Kolassa (2016) analyze a a crossover study
   1. Studies US military subject to physical trauma
   2. Measures efficacy of canine therapy on PTSD, vs standard therapy.
   3. Performance was evaluated on both therapies for subject j: $X_j^i = 1$ if canine therapy superior for subject j with ordering $i \in \{1, 2\}$.
2. Hypotheses:
   1. Null hypothesis is that there is no systematic difference in therapies, regardless of ordering.
   2. Alternative hypothesis is that canine therapy is superior for at least one of the ordering.
3. p-value is given by $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1, \bar{X}^2 \geq \bar{x}^2\right]$.

Notation

1. Random vector $\boldsymbol{X}_i = (X_i^1, \ldots, X_i^p)$ has a continuous distribution,

Notation

1. Random vector $\boldsymbol{X}_i = (X_i^1, \ldots, X_i^p)$ has a continuous distribution,
2. Moment generating function

$$M(\tau_1, \ldots, \tau_p) = \mathrm{E}\left[\exp(\tau_1 X_i^1 + \cdots + \tau_p X_i^p)\right],$$

Notation

1. Random vector $\boldsymbol{X}_i = (X_i^1, \ldots, X_i^p)$ has a continuous distribution,
2. Moment generating function

$$M(\tau_1, \ldots, \tau_p) = \mathrm{E}\left[\exp(\tau_1 X_i^1 + \cdots + \tau_p X_i^p)\right],$$

    1. finite for $|\tau_i| < \epsilon$ for all $i \leq p$ and some $\epsilon > 0$.

Notation

1. Random vector $\boldsymbol{X}_i = (X_i^1, \ldots, X_i^p)$ has a continuous distribution,
2. Moment generating function

$$M(\tau_1, \ldots, \tau_p) = \mathrm{E}\left[\exp(\tau_1 X_i^1 + \cdots + \tau_p X_i^p)\right],$$

   1. finite for $|\tau_i| < \epsilon$ for all $i \leq p$ and some $\epsilon > 0$.
3. Let the cumulant generating function be $K(\tau_1, \ldots, \tau_p) = \log(M(\tau_1, \ldots, \tau_p))$.

Notation

1. Random vector $\boldsymbol{X}_i = (X_i^1, \dots, X_i^p)$ has a continuous distribution,
2. Moment generating function

$$M(\tau_1, \dots, \tau_p) = \mathrm{E}\left[\exp(\tau_1 X_i^1 + \cdots + \tau_p X_i^p)\right],$$

   1. finite for $|\tau_i| < \epsilon$ for all $i \leq p$ and some $\epsilon > 0$.
3. Let the cumulant generating function be $K(\tau_1, \dots, \tau_p) = \log(M(\tau_1, \dots, \tau_p))$.
4. Let $\hat{\tau}_j$ satisfy $K^k(\tau_1, \dots, \tau_p) = \bar{x}^k$ for $k = 1, \dots, p$.

Notation

1. Random vector $\boldsymbol{X}_i = (X_i^1, \ldots, X_i^p)$ has a continuous distribution,
2. Moment generating function

$$M(\tau_1, \ldots, \tau_p) = \mathrm{E}\left[\exp(\tau_1 X_i^1 + \cdots + \tau_p X_i^p)\right],$$

    1. finite for $|\tau_i| < \epsilon$ for all $i \leq p$ and some $\epsilon > 0$.
3. Let the cumulant generating function be $K(\tau_1, \ldots, \tau_p) = \log(M(\tau_1, \ldots, \tau_p))$.
4. Let $\hat{\tau}_j$ satisfy $K^k(\tau_1, \ldots, \tau_p) = \bar{x}^k$ for $k = 1, \ldots, p$.
5. Let $\bar{\boldsymbol{X}} = \sum_{i=1}^n \boldsymbol{X}_i / n$

Objective: Approximate joint tail probabilities for $\bar{X}$

1. of n independent and identically distributed random vectors $(X_i^1, \ldots, X_i^p)$.

Objective: Approximate joint tail probabilities for $\bar{X}$

1. of n independent and identically distributed random vectors $(X_i^1, \ldots, X_i^p)$.
2. Use existence of higher moments to obtain better performance than CLT.

Objective: Approximate joint tail probabilities for $\bar{X}$

1. of n independent and identically distributed random vectors $(X_i^1, \ldots, X_i^p)$.
2. Use existence of higher moments to obtain better performance than CLT.
    1. Gives Edgeworth series.

Objective: Approximate joint tail probabilities for $\bar{X}$

1. of n independent and identically distributed random vectors $(X_i^1, \ldots, X_i^p)$.
2. Use existence of higher moments to obtain better performance than CLT.
    1. Gives Edgeworth series.
3. Use existence of cumulant generating function to obtain better relative error behavior than Edgeworth series.

Objective: Approximate joint tail probabilities for $\bar{X}$

1. of n independent and identically distributed random vectors $(X_i^1, \ldots, X_i^p)$.
2. Use existence of higher moments to obtain better performance than CLT.
    1. Gives Edgeworth series.
3. Use existence of cumulant generating function to obtain better relative error behavior than Edgeworth series.
    1. Measure theoretical error behavior in terms of inverse powers of $\sqrt{n}$.

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \, d\tau_1 \cdots d\tau_p$$

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \, d\tau_1 \cdots d\tau_p$$
    1. $\oint = \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_p - i\infty}^{c_p + i\infty}$

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \; d\tau_1 \cdots d\tau_p$$
    1. $\oint = \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_p - i\infty}^{c_p + i\infty}$
    2. $\boldsymbol{c} = (c_1, \ldots, c_p)$ in the interior of the domain of K.

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\, d\tau_1 \cdots d\tau_p$$
    1. $\oint = \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_p - i\infty}^{c_p + i\infty}$
    2. $\boldsymbol{c} = (c_1, \ldots, c_p)$ in the interior of the domain of K.
2. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right] = \mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1 \cap \ldots \cap \bar{X}^p \geq \bar{x}^p\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\, \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\ d\tau_1 \cdots d\tau_p$$
   1. $\oint = \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_p - i\infty}^{c_p + i\infty}$
   2. $\boldsymbol{c} = (c_1, \ldots, c_p)$ in the interior of the domain of K.
2. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right] = \mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1 \cap \ldots \cap \bar{X}^p \geq \bar{x}^p\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\ \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
   1. $c_i > 0$ for all i

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\; d\tau_1 \cdots d\tau_p$$
    1. $\oint = \int_{c_1 - i\infty}^{c_1 + i\infty} \cdots \int_{c_p - i\infty}^{c_p + i\infty}$
    2. $\boldsymbol{c} = (c_1, \ldots, c_p)$ in the interior of the domain of K.
2. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right] = \mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1 \cap \ldots \cap \bar{X}^p \geq \bar{x}^p\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\; \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
    1. $c_i > 0$ for all i
3. When K is exactly quadratic, get Gaussian tail probabilities, exactly.

Inversion Integrals:

1. Density $f_n(\bar{x}^1, \ldots, \bar{x}^p)$ is
$$\frac{n^p}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\ d\tau_1 \cdots d\tau_p$$
    1. $\oint = \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_p-i\infty}^{c_p+i\infty}$
    2. $\boldsymbol{c} = (c_1, \ldots, c_p)$ in the interior of the domain of K.
2. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right] = \mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1 \cap \ldots \cap \bar{X}^p \geq \bar{x}^p\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i])\ \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
    1. $c_i > 0$ for all i
3. When K is exactly quadratic, get Gaussian tail probabilities, exactly.
    1. Write $\bar{\Phi}(\boldsymbol{z}) = \mathrm{P}\left[Z^1 \geq z^1 \cap \ldots \cap Z^p \geq z^p\right]$ for Z_i independent standard normals.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

   $$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

   $$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

$$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.
3. Exponentiate cubic and quartic terms.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

   $$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.
3. Exponentiate cubic and quartic terms.
4. Integrate term-wise.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

$$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.
3. Exponentiate cubic and quartic terms.
4. Integrate term-wise.
5. Then $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1\right]$ is approximately

$$\exp(n[\hat{z}^2 - \hat{\omega}^2]) \left[\bar{\Phi}(\sqrt{n}\hat{z}) \left(1 - n\frac{\hat{\rho}_3}{6}\right) + \phi(\sqrt{n}\hat{z}) \frac{\hat{\rho}^3(n\hat{z}^2 - 1)}{6\sqrt{n}}\right].$$

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

$$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.
3. Exponentiate cubic and quartic terms.
4. Integrate term-wise.
5. Then $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1\right]$ is approximately

$$\exp(n[\hat{z}^2 - \hat{\omega}^2])\left[\bar{\Phi}(\sqrt{n}\hat{z})\left(1 - n\frac{\hat{\rho}_3}{6}\right) + \phi(\sqrt{n}\hat{z})\frac{\hat{\rho}^3(n\hat{z}^2 - 1)}{6\sqrt{n}}\right].$$

   1. $\hat{z} = \hat{\tau}\sqrt{K''(\hat{\tau})}$

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

$$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.
3. Exponentiate cubic and quartic terms.
4. Integrate term-wise.
5. Then $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1\right]$ is approximately

$$\exp(n[\hat{z}^2 - \hat{\omega}^2])\left[\bar{\Phi}(\sqrt{n}\hat{z})\left(1 - n\frac{\hat{\rho}_3}{6}\right) + \phi(\sqrt{n}\hat{z})\frac{\hat{\rho}^3(n\hat{z}^2 - 1)}{6\sqrt{n}}\right].$$

    1. $\hat{z} = \hat{\tau}\sqrt{K''(\hat{\tau})}$
    2. $\hat{\omega} = \mathrm{sign}(\hat{\tau})\sqrt{2(\hat{\tau}\bar{x} - K(\hat{\tau})}$

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Expand $K(\tau) - \tau\bar{x}$ about $\hat{\tau}$:

$$K(\hat{\tau}) - \hat{\tau}\bar{x} + K''(\hat{\tau})(\tau - \hat{\tau})^2/2 + K'''(\hat{\tau})(\hat{\tau} - \hat{\tau})^3/6 + K''''(\tau^?)(\hat{\tau} - \hat{\tau})^4/24.$$

2. Set $c_1 = \hat{\tau}$.
3. Exponentiate cubic and quartic terms.
4. Integrate term-wise.
5. Then $\mathrm{P}\left[\bar{X}^1 \geq \bar{x}^1\right]$ is approximately

$$\exp(n[\hat{z}^2 - \hat{\omega}^2])\left[\bar{\Phi}(\sqrt{n}\hat{z})\left(1 - n\frac{\hat{\rho}_3}{6}\right) + \phi(\sqrt{n}\hat{z})\frac{\hat{\rho}^3(n\hat{z}^2-1)}{6\sqrt{n}}\right].$$

   1. $\hat{z} = \hat{\tau}\sqrt{K''(\hat{\tau})}$
   2. $\hat{\omega} = \mathrm{sign}(\hat{\tau})\sqrt{2(\hat{\tau}\bar{x} - K(\hat{\tau})}$
   3. $\hat{\rho}_3 = K'''(\hat{\tau})K''(\hat{\tau})^{-3/2}$.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Re-parameterize the inversion integral in terms of ω satisfying $(\omega - \hat{\omega})^2/2 = K(\tau) - \tau x - K(\hat{\tau}) + \hat{\tau} x$.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Re-parameterize the inversion integral in terms of ω satisfying $(\omega - \hat{\omega})^2/2 = K(\tau) - \tau x - K(\hat{\tau}) + \hat{\tau} x$.
2. Integral is $\frac{1}{(2\pi i)} \oint \exp(n[\omega^2/2 - \hat{\omega}\omega])\lambda \frac{d\omega}{\omega}$.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Re-parameterize the inversion integral in terms of ω satisfying $(\omega - \hat{\omega})^2/2 = K(\tau) - \tau x - K(\hat{\tau}) + \hat{\tau} x$.
2. Integral is $\frac{1}{(2\pi i)} \oint \exp(n[\omega^2/2 - \hat{\omega}\omega])\lambda \frac{d\omega}{\omega}$.
    1. $\lambda = \frac{\omega}{\tau} \frac{d\tau}{d\omega}$.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Re-parameterize the inversion integral in terms of ω satisfying $(\omega - \hat{\omega})^2/2 = K(\tau) - \tau x - K(\hat{\tau}) + \hat{\tau} x$.
2. Integral is $\frac{1}{(2\pi i)} \oint \exp(n[\omega^2/2 - \hat{\omega}\omega])\lambda \frac{d\omega}{\omega}$.
    1. $\lambda = \frac{\omega}{\tau} \frac{d\tau}{d\omega}$.
3. Tail probability approximation is $\bar{\Phi}(\hat{\omega}) + \phi(\hat{\omega})(1/\hat{\omega} - 1/\hat{z})/\sqrt{n}$.

One Dimensional Methods: Hao and Kolassa (2016) approach

1. Re-parameterize the inversion integral in terms of ω satisfying $(\omega - \hat{\omega})^2/2 = K(\tau) - \tau x - K(\hat{\tau}) + \hat{\tau} x$.
2. Integral is $\frac{1}{(2\pi i)} \oint \exp(n[\omega^2/2 - \hat{\omega}\omega])\lambda \frac{d\omega}{\omega}$.
    1. $\lambda = \frac{\omega}{\tau} \frac{d\tau}{d\omega}$.
3. Tail probability approximation is $\bar{\Phi}(\hat{\omega}) + \phi(\hat{\omega})(1/\hat{\omega} - 1/\hat{z})/\sqrt{n}$.
    1. Holds without regards to $\hat{\tau} > 0$.

Methods in Multiple Dimensions

1. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \; \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$

Methods in Multiple Dimensions

1. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \; \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
2. Hao and Kolassa (2016) works analogously as with Robinson:

Methods in Multiple Dimensions

1. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \; \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
2. Hao and Kolassa (2016) works analogously as with Robinson:
   1. Expand $K(\tau) - \sum_j \tau_j \bar{x}^j$ about $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_p)$ to get:

$$K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j \bar{x}^j + \sum_{j,k} K^{jk}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)/2$$
$$+ \sum_{j,k,\ell} K^{jk\ell}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)/6$$
$$+ \sum_{j,k,\ell,m} K^{jk\ell m}(\tau^?)(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)(\tau_m - \hat{\tau}_m)/24.$$

Methods in Multiple Dimensions

1. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \; \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
2. Hao and Kolassa (2016) works analogously as with Robinson:
   1. Expand $K(\tau) - \sum_j \tau_j \bar{x}^j$ about $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_p)$ to get:
$$\begin{aligned} K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j \bar{x}^j + \sum_{j,k} K^{jk}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)/2 \\ + \sum_{j,k,\ell} K^{jk\ell}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)/6 \\ + \sum_{j,k,\ell,m} K^{jk\ell m}(\tau^?)(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)(\tau_m - \hat{\tau}_m)/24. \end{aligned}$$
   2. Expand exp of cubic and quartic terms.

Methods in Multiple Dimensions

1. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \ \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
2. Hao and Kolassa (2016) works analogously as with Robinson:
   1. Expand $K(\tau) - \sum_j \tau_j \bar{x}^j$ about $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_p)$ to get:
$$\begin{aligned} K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j \bar{x}^j + \sum_{j,k} K^{jk}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)/2 \\ + \sum_{j,k,\ell} K^{jk\ell}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)/6 \\ + \sum_{j,k,\ell,m} K^{jk\ell m}(\tau^?)(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)(\tau_m - \hat{\tau}_m)/24. \end{aligned}$$
   2. Expand exp of cubic and quartic terms.
   3. Integral of resulting quartic term is $O(1/n)$.

Methods in Multiple Dimensions

1. The tail probability $\mathrm{P}\left[\bar{\boldsymbol{X}} \geq \bar{\boldsymbol{x}}\right]$ is
$$\frac{1}{(2\pi i)^p} \oint \exp(n[K(\tau_1, \ldots, \tau_p) - \sum_{i=1}^{p} \bar{x}^i \tau_i]) \; \frac{d\tau_1 \cdots d\tau_p}{\tau_1 \cdots \tau_p}.$$
2. Hao and Kolassa (2016) works analogously as with Robinson:
    1. Expand $K(\tau) - \sum_j \tau_j \bar{x}^j$ about $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_p)$ to get:
    $$\begin{aligned} K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j \bar{x}^j + \sum_{j,k} K^{jk}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)/2 \\ + \sum_{j,k,\ell} K^{jk\ell}(\hat{\boldsymbol{\tau}})(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)/6 \\ + \sum_{j,k,\ell,m} K^{jk\ell m}(\tau^?)(\tau_j - \hat{\tau}_j)(\tau_k - \hat{\tau}_k)(\tau_\ell - \hat{\tau}_\ell)(\tau_m - \hat{\tau}_m)/24. \end{aligned}$$
    2. Expand exp of cubic and quartic terms.
    3. Integral of resulting quartic term is $O(1/n)$.
    4. Integrate terms up to cubic term-wise.

Downsides of existing approximation

1. Lots of terms

Downsides of existing approximation

1. Lots of terms
2. Terms are not particularly interpretable.

Downsides of existing approximation

1. Lots of terms
2. Terms are not particularly interpretable.
   1. (Multivariate) normal tail probability is multiplied by an exponential factor.

Downsides of existing approximation

1. Lots of terms
2. Terms are not particularly interpretable.
    1. (Multivariate) normal tail probability is multiplied by an exponential factor.
3. You are stuck with $\hat{\tau}_j > 0$

Downsides of existing approximation

1. Lots of terms
2. Terms are not particularly interpretable.
    1. (Multivariate) normal tail probability is multiplied by an exponential factor.
3. You are stuck with $\hat{\tau}_j > 0$
    1. Get around this using Boole's Law

Lugannani and Rice Analog

1. New parameterization $\sum_j \hat{\omega}_j^2/2 = K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j$, $\sum_j(\omega_j - \hat{\omega}_j)^2/2 = K(\boldsymbol{\tau}) - \sum_j \tau_j - (K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j)$

Lugannani and Rice Analog

1. New parameterization $\sum_j \hat{\omega}_j^2/2 = K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j$, $\sum_j(\omega_j - \hat{\omega}_j)^2/2 = K(\boldsymbol{\tau}) - \sum_j \tau_j - (K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j)$
2. Integral is

$$\frac{1}{(2\pi i)^p} \oint \exp(n \sum_j \omega_j^2 - \sum_{i=1}^{p} \hat{\omega}_j \omega_j) \lambda \, \frac{d\omega_1 \cdots d\omega_p}{\omega_1 \cdots \omega_p}.$$

Lugannani and Rice Analog

1. New parameterization $\sum_j \hat\omega_j^2/2 = K(\hat{\boldsymbol\tau}) - \sum_j \hat\tau_j$, $\sum_j(\omega_j - \hat\omega_j)^2/2 = K(\boldsymbol\tau) - \sum_j \tau_j - (K(\hat{\boldsymbol\tau}) - \sum_j \hat\tau_j)$
2. Integral is

$$\frac{1}{(2\pi i)^p}\oint \exp(n\sum_j \omega_j^2 - \sum_{i=1}^p \hat\omega_j\omega_j)\lambda\ \frac{d\omega_1\cdots d\omega_p}{\omega_1\cdots\omega_p}.$$

   1. $\lambda = \frac{\omega_1\cdots\omega_p}{\tau_1\cdots\tau_p}\frac{d\tau_1\cdots d\tau_p}{d\omega_1\cdots d\omega_p}$.

Lugannani and Rice Analog

1. New parameterization $\sum_j \hat{\omega}_j^2/2 = K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j$, $\sum_j(\omega_j - \hat{\omega}_j)^2/2 = K(\boldsymbol{\tau}) - \sum_j \tau_j - (K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j)$
2. Integral is

$$\frac{1}{(2\pi i)^p} \oint \exp(n \sum_j \omega_j^2 - \sum_{i=1}^{p} \hat{\omega}_j \omega_j) \lambda \, \frac{d\omega_1 \cdots d\omega_p}{\omega_1 \cdots \omega_p}.$$

   1. $\lambda = \frac{\omega_1 \cdots \omega_p}{\tau_1 \cdots \tau_p} \frac{d\tau_1 \cdots d\tau_p}{d\omega_1 \cdots d\omega_p}$.
      1. In one dimension, zeros in denominator of λ correspond to zeros in the numerator.

Lugannani and Rice Analog

1. New parameterization $\sum_j \hat{\omega}_j^2/2 = K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j$, $\sum_j (\omega_j - \hat{\omega}_j)^2/2 = K(\boldsymbol{\tau}) - \sum_j \tau_j - (K(\hat{\boldsymbol{\tau}}) - \sum_j \hat{\tau}_j)$
2. Integral is

$$\frac{1}{(2\pi i)^p} \oint \exp(n \sum_j \omega_j^2 - \sum_{i=1}^p \hat{\omega}_j \omega_j) \lambda \ \frac{d\omega_1 \cdots d\omega_p}{\omega_1 \cdots \omega_p}.$$

   1. $\lambda = \frac{\omega_1 \cdots \omega_p}{\tau_1 \cdots \tau_p} \frac{d\tau_1 \cdots d\tau_p}{d\omega_1 \cdots d\omega_p}$.
      1. In one dimension, zeros in denominator of λ correspond to zeros in the numerator.
      2. This fails for $p > 1$.

Re-parameterize to fix lack of alignment of zeros

1. Re-parameterize to $\xi_j = \omega_j - \sum_{k<j} \omega_k \zeta_j^k(\omega_1, \ldots, \omega_{j-1})$ so that $\xi_j = 0$ if and only if $\tau_j = 0$.

Re-parameterize to fix lack of alignment of zeros

1. Re-parameterize to $\xi_j = \omega_j - \sum_{k<j} \omega_k \zeta_j^k(\omega_1, \ldots, \omega_{j-1})$ so that $\xi_j = 0$ if and only if $\tau_j = 0$.
    1. Ex. $\xi_1 = \omega_1$, $\xi_2 = \omega_2 - \omega_1 \zeta_2^1(\omega_1)$ for $\zeta_1^2 = \omega_2(\tau_1, 0)/\omega_1$.

Re-parameterize to fix lack of alignment of zeros

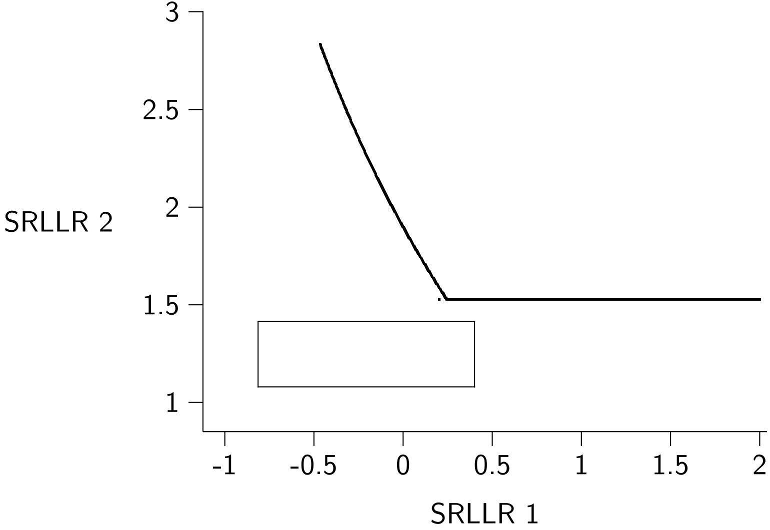
1. Re-parameterize to $\xi_j = \omega_j - \sum_{k<j} \omega_k \zeta_j^k(\omega_1, \ldots, \omega_{j-1})$ so that $\xi_j = 0$ if and only if $\tau_j = 0$.
    1. Ex. $\xi_1 = \omega_1$, $\xi_2 = \omega_2 - \omega_1 \zeta_2^1(\omega_1)$ for $\zeta_1^2 = \omega_2(\tau_1, 0)/\omega_1$.
    2. Adjustment also makes $\{\bar{X}^1 \geq \bar{x}^1 \cap \ldots, \bar{X}^p \geq \bar{x}^p\} \approx \{\hat{\Xi}^1 \geq \hat{\xi}^1 \cap \ldots, \hat{\Xi}^p \geq \hat{\xi}^p\}$.

Re-parameterize to fix lack of alignment of zeros

1. Re-parameterize to $\xi_j = \omega_j - \sum_{k<j} \omega_k \zeta_j^k(\omega_1, \ldots, \omega_{j-1})$ so that $\xi_j = 0$ if and only if $\tau_j = 0$.
   1. Ex. $\xi_1 = \omega_1$, $\xi_2 = \omega_2 - \omega_1 \zeta_2^1(\omega_1)$ for $\zeta_1^2 = \omega_2(\tau_1, 0)/\omega_1$.
   2. Adjustment also makes $\{\bar{X}^1 \geq \bar{x}^1 \cap \ldots, \bar{X}^p \geq \bar{x}^p\} \approx \{\hat{\Xi}^1 \geq \hat{\xi}^1 \cap \ldots, \hat{\Xi}^p \geq \hat{\xi}^p\}$.
   3. Adjustment makes singularities in λ removable.

Adjustment induces sample space rotation for $\hat{\Omega}_j$

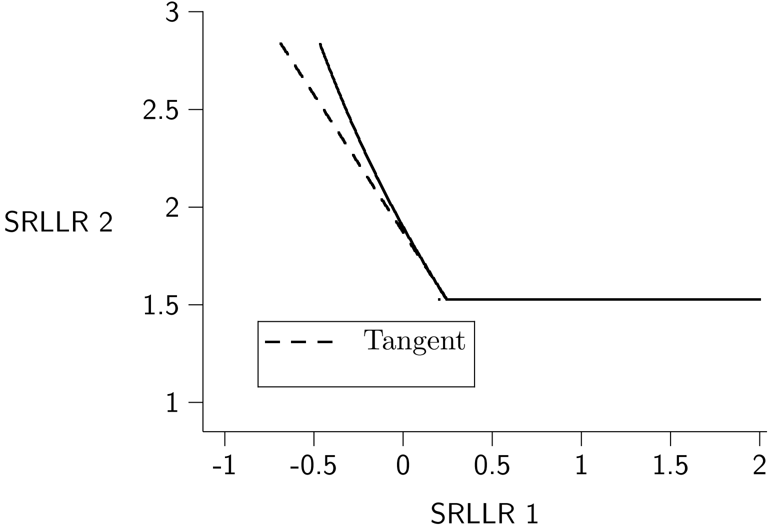
1. using $\zeta_k^j(\hat{\Omega})$.



Correlated gamma example, $\boldsymbol{x} = (4, 5)$

Adjustment induces sample space rotation for $\hat{\Omega}_j$

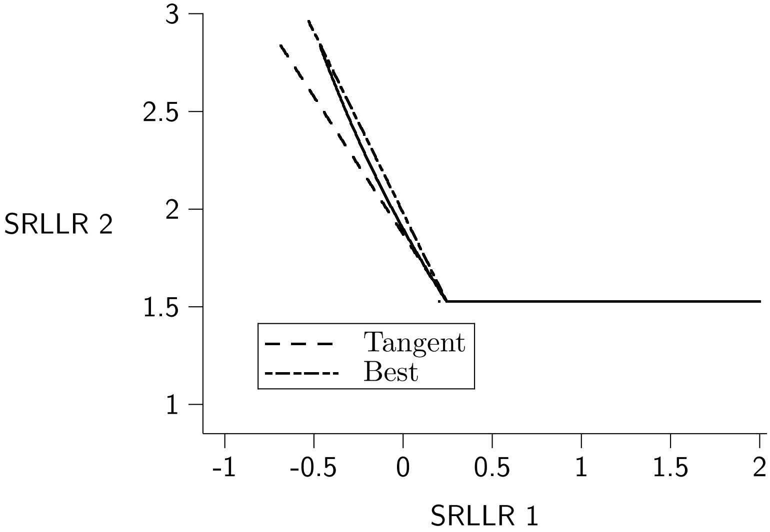
1. using $\zeta_k^j(\hat{\Omega})$.



Correlated gamma example, $\boldsymbol{x} = (4, 5)$

Adjustment induces sample space rotation for $\hat{\Omega}_j$

1. using $\zeta_k^j(\hat{\Omega})$.



Correlated gamma example, $\boldsymbol{x} = (4, 5)$

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
$\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}; \boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j}; \boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
   $\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}};\boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j};\boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$
   1. $\boldsymbol{\Sigma}$ is generated by transformation from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$.

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
$\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}; \boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j}; \boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$
    1. $\boldsymbol{\Sigma}$ is generated by transformation from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$.
    2. σ_j is a standard error for $\hat{\tau}_j$.

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
$\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}};\boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j};\boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$
    1. $\boldsymbol{\Sigma}$ is generated by transformation from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$.
    2. σ_j is a standard error for $\hat{\tau}_j$.
2. In order to make λ have removable singularities,

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
   $\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}; \boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j}; \boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$
   1. $\boldsymbol{\Sigma}$ is generated by transformation from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$.
   2. σ_j is a standard error for $\hat{\tau}_j$.
2. In order to make λ have removable singularities,
   1. exponent in inversion integral is no longer exactly quadratic.

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
$\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}; \boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j}; \boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$
    1. $\boldsymbol{\Sigma}$ is generated by transformation from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$.
    2. σ_j is a standard error for $\hat{\tau}_j$.
2. In order to make λ have removable singularities,
    1. exponent in inversion integral is no longer exactly quadratic.
    2. C adjusts for this.

New approximation is simpler

1. $\mathrm{P}\left[X^1 \geq x^1 \cap \ldots \cap X^p \geq x^p\right] \approx$
   $\bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}; \boldsymbol{\Sigma}) + \sum_{j=1}^{p} \bar{\Phi}(\sqrt{n}\hat{\boldsymbol{\xi}}_{-j}; \boldsymbol{\Sigma}_j)\phi(\sqrt{n}\xi_j)(1/\xi_j - 1/(\hat{\tau}_j/\sigma_j)) + C.$
   1. $\boldsymbol{\Sigma}$ is generated by transformation from $\boldsymbol{\omega}$ to $\boldsymbol{\xi}$.
   2. σ_j is a standard error for $\hat{\tau}_j$.
2. In order to make λ have removable singularities,
   1. exponent in inversion integral is no longer exactly quadratic.
   2. C adjusts for this.
3. Avoids requirement that $\hat{\tau}_j > 0$.

Example

1. $p = 2$,

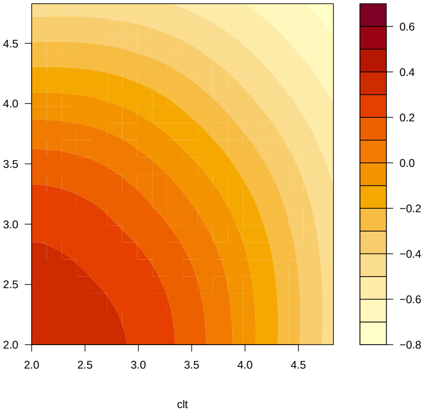
Example

1. $p = 2$,
2. $X_1 = Z_1 + Z_2$, $X_2 = Z_1 + Z_3$, Z_j independent exponentials.

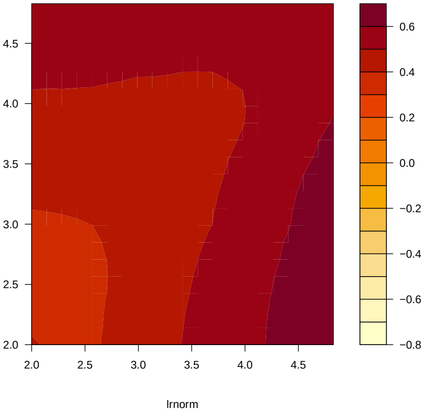
Example

1. $p = 2$,
2. $X_1 = Z_1 + Z_2$, $X_2 = Z_1 + Z_3$, Z_j independent exponentials.
3. $n = 1$!

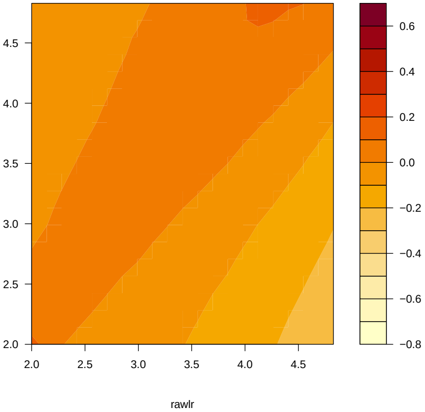
Relative Error: Normal on raw scale



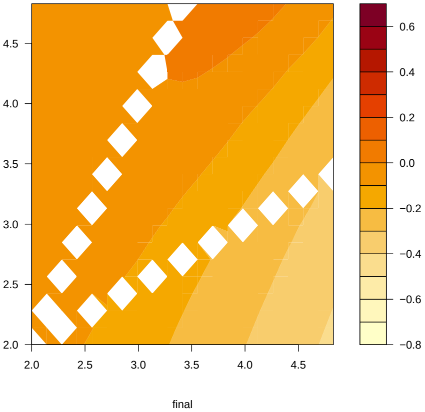
Relative Error: Normal on SRLLR scale



Relative Error: SRLLR scale, adjust for marginal non-normality



Relative Error: SRLLR scale, all adjustments



Work in various stages of completion

1. Addressing removable singularities in C.

Work in various stages of completion

1. Addressing removable singularities in C.
2. $p > 2$.

Work in various stages of completion

1. Addressing removable singularities in C.
2. $p > 2$.
3. r^* version placing $O(1/\sqrt{n})$ corrections into argument of $\bar{\Phi}$.

Work in various stages of completion

1. Addressing removable singularities in C.
2. $p > 2$.
3. r^* version placing $O(1/\sqrt{n})$ corrections into argument of $\bar{\Phi}$.
4. Extension to lattice variables

Work in various stages of completion

1. Addressing removable singularities in C.
2. $p > 2$.
3. r^* version placing $O(1/\sqrt{n})$ corrections into argument of $\bar{\Phi}$.
4. Extension to lattice variables
5. Extension to conditional distributions.

Hao, Y. and Kolassa, J. (2016). Multiple hypotheses in the analysis of a crossover trial. *Statistics in Biosciences*, 8(2):253–263. Funding Information: The second author was supported by Grant NSF DMS 0906569. Publisher Copyright: © 2016, International Chinese Statistical Association.

Hao, Y. and Kolassa, J. (2016). Multiple hypotheses in the analysis of a crossover trial. *Statistics in Biosciences*, 8(2):253–263. Funding Information: The second author was supported by Grant NSF DMS 0906569. Publisher Copyright: © 2016, International Chinese Statistical Association.

Kolassa, J. (2003). Multivariate saddlepoint tail probability approximations. *Annals of Statistics*, 31:274–286.

Hao, Y. and Kolassa, J. (2016). Multiple hypotheses in the analysis of a crossover trial. *Statistics in Biosciences*, 8(2):253–263. Funding Information: The second author was supported by Grant NSF DMS 0906569. Publisher Copyright: © 2016, International Chinese Statistical Association.

Kolassa, J. (2003). Multivariate saddlepoint tail probability approximations. *Annals of Statistics*, 31:274–286.

Lugannani, R. and Rice, S. (1980). Saddle point approximation for the distribution of the sum of independent random variables. *Advances in Applied Probability*, 12:475–490.

Hao, Y. and Kolassa, J. (2016). Multiple hypotheses in the analysis of a crossover trial. *Statistics in Biosciences*, 8(2):253–263. Funding Information: The second author was supported by Grant NSF DMS 0906569. Publisher Copyright: © 2016, International Chinese Statistical Association.

Kolassa, J. (2003). Multivariate saddlepoint tail probability approximations. *Annals of Statistics*, 31:274–286.

Lugannani, R. and Rice, S. (1980). Saddle point approximation for the distribution of the sum of independent random variables. *Advances in Applied Probability*, 12:475–490.

Robinson, J. (1982). Saddlepoint approximations for permutation tests and confidence intervals. *Journal of the Royal Statistical Society Series B*, 44:91–101.