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Saddlepoint Approximations for Permutation Tests and Confidence Intervals

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Summary

Asymptotic approximations for the tail probabilities of certain statistics with complex moment generating functions are obtained by the usual methods of large deviation theory. It is pointed out that this approximation should be adequate over the entire range but of particular use for extreme tails. Approximations to the significance levels for one and two sample permutation tests are obtained and these tests are inverted to obtain approximate confidence intervals. These methods are applied to some numerical examples and the results are compared with the exact values and with approximations obtained from Edgeworth expansions.

Keywords: ASYMPTOTIC APPROXIMATIONS; CONFIDENCE INTERVALS; EDGEWORTH EXPANSIONS; LARGE DEVIATIONS; PERMUTATION TESTS; SADDLEPOINT METHODS

1. Introduction

There are computational difficulties inherent in the use of permutation tests unless good approximations are available for tail probabilities. Edgeworth expansions have been obtained by Albers, Bickel and Van Zwet (1976), Bickel and Van Zwet (1978) and Robinson (1978), and these give good approximations in most cases. However, it might be expected that saddlepoint methods, or large deviation results, might give better approximations, especially in the tails, and it is there that the approximations are most important.

Saddlepoint approximations have been considered by Daniels (1954), Blackwell and Hodges (1959) and, more recently, by Barndorff-Nielsen and Cox (1979). These authors considered saddlepoint approximations for densities by expanding the complex moment generating function in a Taylor series about a saddlepoint of that function. Bahadur and Ranga Rao (1960) and Petrov (1965) considered related approximations for tail probabilities. All these papers deal only with sums of independent, identically distributed random variables. In Section 2, we will consider how these methods may be used to obtain approximations for tail probabilities in a slightly more general situation, when we can obtain a normal approximation for an exponentially shifted distribution. Further, if one more term of an Edgeworth expansion is available, we show how to obtain a second saddlepoint approximation with a smaller relative error. It is pointed out that this approximation is equivalent to the numerically integrated density approximations of Daniels (1954).

We will describe how these methods may be used to give approximations for the significance levels of permutation tests in the one and two sample problems although the statistics in these cases are not sums of independent and identically distributed random variables. It will also be shown that these approximations can readily be obtained numerically and that then the tests can be inverted numerically to give approximate confidence intervals. Finally, we will consider some examples from the literature and compare the approximations obtained by these methods with Edgeworth approximations and with exact values.

Proofs of validity of the expansions by obtaining inequalities on error terms are quite complicated but do not involve any new techniques, so these are only outlined here with references to related work.

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2. THE SADDLEPOINT APPROXIMATION

Let T_n be a statistic with distribution function F_n. For any fixed value u, let

$$V_n(x) = [Q_n(u)]^{-1} \int_{-\infty}^{x} e^{uy}\, dF_n(y),$$

where

$$Q_n(u) = \int_{-\infty}^{\infty} e^{uy}\, dF_n(y),$$

is assumed to exist in the range $0<u<B_n \leqslant \infty$, be an exponentially shifted distribution function. If $V_n(x)$ can be uniformly approximated by $G_n(x) = \Phi(\sigma_n^{-1}(x-m_n))$, where Φ denotes the distribution function of a standard normal variate and $m_n = m_n(u)$, $\sigma_n^2 = \sigma_n^2(u)$ are the mean and variance of V_n, then we can use this approximation to obtain approximations for $1-F_n(x_n)$ and $F_n(-x_n)$, for any $x_n>0$. For

$$\begin{aligned} 1-F_n(x_n) &= Q_n(u) \int_{x_n}^{\infty} e^{-uy}\, dV_n(y) \\ &= (2\pi\sigma_n^2)^{-\frac{1}{2}}\, Q_n(u) \int_{x_n}^{\infty} \exp(-uy-(y-m_n)^2/2\sigma_n^2)\, dy \\ &\quad + Q_n(u) \int_{x_n}^{\infty} e^{-uy}\, d(V_n(y)-G_n(y)) \\ &= A_1 + A_2. \end{aligned}$$

As shown, for example, in Petrov (1965), we can choose u such that $m_n(u) = x_n$, whenever x_n is such that $1-F_n(x_n)>0$, then we obtain

$$A_1 = Q_n(u) \exp(-um_n + \tfrac{1}{2}u^2\sigma_n^2)(1-\Phi(u\sigma_n)). \tag{1}$$

We will call this the first saddlepoint approximation.

Now if

$$\sup_x | V_n(x) - G_n(x) | < R_n,$$

then using integration by parts we get

$$\begin{aligned} |A_2| &= Q_n(u) \left| \exp(-um_n)(V_n(m_n)-G_n(m_n)) \right. \\ &\qquad \left. + \int_{m_n}^{\infty} (V_n(y)-G_n(y))\, u\, e^{-uy}\, dy \right| \\ &\leqslant 2Q_n(u) \exp(-um_n)\, R_n, \end{aligned} \tag{2}$$

where again u has been chosen so that $m_n(u) = x_n$.

We need a result of the type of a Berry–Esseen inequality, to ensure that R_n is small enough and this will be shown to be the case in the examples considered here. Generally, we need to show that R_n is $O(n^{-\frac{1}{2}})$. Then by noting that

$$\log Q_n(u) - um_n < 0$$

and, in particular, is large and negative when x_n is large, we see that the saddlepoint approximation should be as good as a normal approximation throughout the range and should be much better in the extreme tails.

It is worthwhile pointing out the relation between this approximation and the results of Cramér (1938) and of Bahadur and Ranga Rao (1960) and Petrov (1965) for sums of independent and identically distributed random variables. In the Cramér case it is shown that for $x_n < \varepsilon\sqrt{n}$, for some $\varepsilon > 0$,

$$u\sigma_n = x_n(1+o(1)), \tag{3}$$

$$\log Q_n(u) - um_n + \tfrac{1}{2}u^2\sigma_n^2 = \frac{x_n^3}{\sqrt{n}}\lambda_n\left(\frac{x_n}{\sqrt{n}}\right), \tag{4}$$

where $\lambda_n(t)$ is a convergent power series with coefficients depending on the moments. This result is concerned with the relative error of the normal approximation, rather than with obtaining an alternative approximation. The results of Bahadur and Ranga Rao and of Petrov are concerned with the case x_n large and they use the further approximation

$$\exp(\tfrac{1}{2}u^2\sigma_n^2)(1-\Phi(u\sigma_n)) = (2\pi)^{-\frac{1}{2}}(u\sigma_n)^{-1}(1+o(1)).$$

It is noted that this approximation is not adequate for the small sample approximations considered here since $u\sigma_n$ is not large enough.

If $V_n(x)$ can be uniformly approximated by an Edgeworth series, then a more accurate result may be obtained by including terms from this, thus increasing the relative accuracy over the whole range. We will consider the inclusion of terms of order $n^{-\frac{1}{2}}$. Suppose

$$G_n(x) = \Phi(y) + \tfrac{1}{2}\kappa_{1n}\phi(y) - \tfrac{1}{6}\kappa_{3n}H_2(y)\phi(y), \tag{5}$$

where $\phi(x) = \Phi'(x)$, $H_2(x)\Phi(x) = \phi''(x)$ and we write $y = (x-m_n)/\sigma_n$. Here κ_{1n} and κ_{3n} are usually of order $n^{-\frac{1}{2}}$ and will be defined in the particular cases later. In the case of sums of i.i.d. random variables, $\kappa_{1n} = 0$ and κ_{3n} is the standardized third moment of V_n. Then

$$\begin{aligned} 1-F_n(x_n) &= Q_n(u)\int_{x_n}^{\infty} e^{-uy}\,dG_n(y) + Q_n(u)\int_{x_n}^{\infty} e^{-uy}\,d(V_n(y)-G_n(y)) \\ &= B_1 + B_2. \end{aligned} \tag{6}$$

Now if we choose u such that $m_n(u) = x_n$, whenever $1-F_n(x_n) > 0$, then integration in B_1 gives

$$B_1 = A_1[1 - \tfrac{1}{2}\kappa_{1n}W_1(u\sigma_n) + \tfrac{1}{6}\kappa_{3n}W_3(u\sigma_n)], \tag{7}$$

where

$$W_1(v) = \frac{\phi(v)}{1-\Phi(v)} - v$$

and

$$W_3(v) = \frac{(v^2-1)\phi(v)}{1-\Phi(v)} - v^3.$$

The proof of this result is given in Appendix 1. We will call B_1 the second saddlepoint approximation. If

$$\sup_x |V_n(x) - G_n(x)| < R'_n,$$

then as in (2), we have

$$|B_2| \leqslant 2Q_n(u)\exp(-um_n)R'_n.$$

If, for example, we can show that $R'_n = O(n^{-1})$, then $B_2 = B_1\,O(n^{-1})$, so the relative error of the approximation B_2 is of order n^{-1}. In the case of sums of i.i.d. random variables, it is sufficient that the variables possess fourth moments and that Cramér's condition (Feller, 1971, p. 541) holds. Saulis (1969) has obtained an expansion involving κ_{3n} when discussing extensions of the results given at (3) and (4).

In Appendix 1 it is also shown that if a saddlepoint approximation is available for the density $f_n(y)$, corresponding to $F_n(y)$, then the integral of this density for $y>x_n$, may be approximated by B_1 with a relative error of $O(n^{-1})$. This is to be expected since both approximations have relative errors of $O(n^{-1})$, but it seems worthwhile to show explicitly how to obtain the relationship between these approximations, considered by Daniels (1954) and Barndorff-Nielsen and Cox (1979), and those given here. However, it should be noted that in the case of the permutation tests considered in Sections 3 and 4, $V_n(x)$ does not have a density, so the local limit theorem could only be used formally. This formal approximation has been suggested, in the case of a one sample permutation test, by Daniels (1955) and in the case of a two sample permutation test, by Daniels (1958).

3. The One-Sample Case

Suppose that we observe the random variables $X_1, \ldots, X_n$. We will consider either the usual model

(1) $X_1, \ldots, X_n$ are i.i.d. with a distribution function $F(x-\theta)$, where $F(y)$ is symmetric;

or a randomization model

(2) $X_1, \ldots, X_n$ are differences of pairs of observations from a randomized trial where the treatment effect is additive; so a model for X_i is

$$X_i = \theta + Y_i U_i,$$

where $Y_1, \ldots, Y_n$ are 'plot errors' and have some relatively arbitrary distribution, or may be considered as fixed values, and $U_1, \ldots, U_n$ take values 1 or -1 with probabilities $\frac{1}{2}$. Of course, the second model includes the first as a special case. We will comment later on the conditions to be imposed on the distribution of $Y_1, \ldots, Y_n$.

In either case, we are interested in the hypothesis H: $\theta = \theta_0$. The appropriate permutation test in this case is conditional on $|a_1|, \ldots, |a_n|$, where

$$a_i = (X_i - \theta_0)/[\sum_i (X_i - \theta_0)^2]^{\frac{1}{2}}.$$

The test statistic is

$$T_n = \sum_k V_k |a_k|,$$

where V_k are independent random variables taking values 1 and -1 with probabilities $\frac{1}{2}$. The observed value is $t_n = \Sigma_k a_k$ and the significance level is $P(T_n \geqslant t_n)$, where P indicates the conditional distribution given $|a_1|, \ldots, |a_n|$. If we wish a two sided test then $|T_n|$ is the test statistic.

In this case the moment generating function is

$$Q_n(z) = \exp[\sum_k K(2za_k)],$$

where

$$K(x) = \log(p\, e^{qx} + q\, e^{-px}),$$

where $p+q=1$, and, in this section, $p=\frac{1}{2}$. If we put $z = u+iv$ and consider a Taylor expansion of $\log Q_n(u+iv)$ about $v=0$, we obtain in a standard fashion,

$$Q_n(z) = \exp[\sum_k K(2ua_k) + ivm_n - \tfrac{1}{2}v^2 \sigma_n^2](1+R),$$

where

$$m_n = m_n(u) = 2\sum_k a_k K'(2ua_k),$$

$$\sigma_n^2 = \sigma_n^2(u) = 4\sum_k a_k^2 K''(2ua_k),$$

and where it can be shown that, if we take B to be a positive constant which may be different at each occurrence,

$$|R| < B\sigma_n^{-3}|v|^3 \exp(\tfrac{1}{4}v^2\sigma_n^2)\sum_k |a_k|^3 K''(2ua_k) \tag{8}$$

for $|v| < B(\Sigma_k |a_k|^3)^{-1}$. Then using the usual Esseen inequality (Feller, 1971, p. 538), we can obtain the result

$$|R_n| = \sup_x |V_n(x) - G_n(x)| < B\sum_k |a_k|^3.$$

The detailed proofs of these results are similar to, but simpler than, those obtained by Robinson (1977) and discussed in the next section.

Now we can apply the methods of Section 2, to obtain a saddlepoint approximation for $P(T_n \geqslant t_n)$. First we need to solve the equation

$$m_n(u) = t_n \tag{9}$$

for u. This must be done numerically, but since $m_n(u)$ is an increasing function this is easily done by an iterative Gauss–Newton method. Then, we can calculate $\sigma_n(u)$ and substitute in (1) to obtain the approximation A_1.

Under the conditions given in (2.15) and (2.16) of Albers, Bickel and Van Zwet (1976), an Edgeworth expansion (5) exists with $\kappa_{1n} = 0$ and

$$\kappa_{3n} = \sum_k a_k^3 K'''(2ua_k)/[\sum_k a_k^2 K''(2ua_k)]^{3/2}.$$

This can be shown by expanding $\log Q_n(z)$ about $z = u$, as above, and taking one further term in the expansion to obtain

$$Q_n(z) = \exp[\sum_k K(2ua_k) + ivm_n - \tfrac{1}{2}v^2\sigma_n^2](1 + \tfrac{1}{6}(iv)^3\sum_k a_k^3 K'''(2ua_k) + R'),$$

where it can be shown by the methods of Robinson (1977, 1978) that

$$|R'| < B\sigma_n^{-4}|v|^4 \exp(\tfrac{1}{4}v^2\sigma_n^2)\sum_k a_k^4,$$

for $|v| < B(\Sigma_k |a_k|^3)^{-1}$. This approximation can be extended to the range $B(\Sigma_k |a_k|^3)^{-1} \leqslant v < B(\Sigma_k a_k^4)^{-1}$, using the condition (2.16) of Albers, Bickel and Van Zwet (1976). Then, since $Q_n(u+iv)/Q_n(u)$ is the characteristic function of $V_n(x)$, we can show that this is approximated by

$$g_n(v) = \exp[ivm_n - \tfrac{1}{2}v^2\sigma_n^2](1 + \tfrac{1}{6}(iv)^3\sum_k a_k^3 K'''(2ua_k)).$$

Now from the usual Esseen inequality (Feller (1971) p. 538) we can show that

$$\sup_x |V_n(x) - G_n(x)| < B\sum_k a_k^4.$$

After solving (9), we can calculate κ_{3n} and hence the second saddlepoint approximation by substituting in (7).

4. The Two-sample Case

Suppose that we observe the random variables $X_1, \ldots, X_m$ and $X_{m+1}, \ldots, X_{m+n}$. Consider either the usual model

(1) $X_1, \ldots, X_m$ and $X_{m+1}, \ldots, X_{m+n}$ are independent and identically distributed with distribution functions $F(x-\theta_1)$ and $F(x-\theta_2)$, respectively;

or a randomization model

(2) $X_1, \ldots, X_m$ are observed on m experimental units chosen randomly from $m+n$ and given one treatment and $X_{m+1}, \ldots, X_{m+n}$ are the observations on the other units which are given another treatment; it is assumed that the treatment effects are additive; so a

model for these random variables is

$$X_i = Z_{R_i} + \theta_1, \quad i = 1, \ldots, m,$$
$$X_{j+m} = Z_{R_{j+m}} + \theta_2, \quad j = 1, \ldots, n,$$

where $Z_1, \ldots, Z_{n+m}$ have some relatively arbitrary distribution and $(R_1, \ldots, R_{m+n})$ is a random vector taking each of the $(n+m)!$ permutations of $(1, \ldots, n+m)$ with equal probability.

In either case, we are interested in the hypothesis H: $\theta_1 - \theta_2 = \delta_0$. The appropriate permutation test in this case is conditional on $a_1, \ldots, a_N$, where $N = n+m$ and

$$a_k = (Y_k - \bar{Y})/[\sum_{i=1}^N (Y_i - \bar{Y})^2]^{\frac{1}{2}},$$

for

$$Y_i = X_i - \delta_0, \quad i = 1, \ldots, m,$$
$$= X_i, \qquad i = m+1, \ldots, N.$$

The test statistic is

$$T_N = w_N^{-\frac{1}{2}} \sum_{i=1}^m a_{S_i},$$

where $w_N = Npq/(N-1)$ and $(S_1, \ldots, S_N)$ is a random vector, independent of all preceding random variables, taking each permutation of $(1, \ldots, N)$ with equal probability. The observed value is $t_N = w_N^{-\frac{1}{2}} \Sigma_{i=1}^m a_i$ and the significance level is $P(T_N \geqslant t_N)$, where P indicates the conditional distribution given $a_1, \ldots, a_N$. Again, if a two-sided test is required, $|T_N|$ is the test statistic.

It has been shown in Robinson (1977) that if $Q_N(u+iv)$ is the complex moment generating function of T_N, then

$$Q_N(u+iv) = ((Npq)^{-1} \sum_k K_k'')^{-\frac{1}{2}} \exp[\sum_k K_k + ivm_N - \tfrac{1}{2}v^2 \sigma_N^2](1+R)$$

where

$$m_N = w_N^{-\frac{1}{2}} \sum_k a_k K_k',$$
$$\sigma_N^2 = w_N^{-1}[\sum_k a_k^2 K_k'' - (\sum_k a_k K_k'')^2 / \sum_k K_k'']$$

and K_k, K_k', K_k'' are $K(x), K'(x), K''(x)$ at $x = ua_k w_N^{-\frac{1}{2}} + \alpha(u)$, where $\alpha(u)$ is the solution of the equation

$$\sum_k K_k' = 0, \tag{10}$$

for each u, and where, for $b_N = \max_k |a_k|$, $|v| < Bb_N^{-1}$,

$$|R| < Bb_N(|v|^3 + B)\exp(\tfrac{1}{4}v^2 \sigma_N^2).$$

Now using the Esseen inequality again it is shown in Robinson (1977) that

$$\sup_x |V_N(x) - G_N(x)| < Bb_N.$$

By using the methods of Höglund (1978) this inequality could be improved to one with $B\Sigma_k |a_k|^3$ on the right-hand side.

Now the methods of Section 2 may be applied to give the saddlepoint approximation for $P(T_N \geqslant t_N)$. It is necessary to solve simultaneously the equations

$$m_N(u) = t_N \quad \text{and} \quad \sum_k K_k' = 0, \tag{11}$$

for u and $\alpha(u)$. This must be done numerically, but since for each u, $\Sigma_k K_k'$ is an increasing function of α and $m_N(u)$ is an increasing function of u, this can be done iteratively. Then we calculate $\sigma_N(u)$ and substitute in (1) to obtain the approximation.

It will be shown in Appendix 2 that taking an extra term in the expansion used in Robinson (1977) leads to the approximation

$$Q_N(u+iv) = (\sum_k K_k''/Npq)^{-\frac{1}{2}} \exp[\sum_k K_k + ivm_N - \tfrac{1}{2}v^2\sigma_N^2] \times (1-\tfrac{1}{2}iv\sigma_N\kappa_{1N}+\tfrac{1}{6}(iv)^3\sigma_N^3\kappa_{3N}+R'),$$

where

$$\sigma_N\kappa_{1N} = \sum_k a_k K_k'''/\sum_k K_k'' - (\sum_k K_k''')(\sum_k a_k K_k'')/(\sum_k K_k'')^2, \tag{12}$$

$$\sigma_N^3\kappa_{3N} = \sum_k a_k^3 K_k''' - 3\sum_k a_k^2 K_k''' H_k + 3\sum_k a_k K_k''' H_k^2 - H_k^3, \tag{13}$$

where

$$H_k = \sum_k a_k K_k''/\sum_k K_k'',$$

and K_k''' is $K'''(x)$ at $x = ua_k w_N^{-\frac{1}{2}}+\alpha(u)$ for u and α obtained as in (10), and where it can be shown by the methods of Robinson (1977, 1978), that

$$|R'| < B\exp(\tfrac{1}{4}v^2\sigma_N^2)\sum_k a_k^4$$

for $|v| < B(\Sigma_k|a_k|^3)^{-1}$. The approximation can be extended to $|v| < B(\Sigma_k a_k^4)^{-1}$, as in Section 2, then using the Esseen inequality we can show that

$$\sup_x|V_N(x)-G_N(x)| < B\sum_k a_k^4.$$

After solving (11), we can calculate κ_{1N} and κ_{3N} and hence the second saddlepoint approximation by substituting in (7).

It is necessary to remark on the conditions under which $\Sigma_k|a_k|^3$ or $\Sigma_k a_k^4$ will be small enough with high enough probability. If $Z_1,\ldots,Z_n$ are i.i.d. with finite third moment and non-zero variance, then $\Sigma_k|a_k|^3$ and $\Sigma_k a_k^4$ will certainly be of order $N^{-\frac{1}{2}}$ and N^{-1} with probability less than one by a quantity of order $N^{-\frac{1}{2}}$ and N^{-1}. Further, the conditions for Edgeworth expansions to exist will require some sort of continuity properties on the distributions of $Z_1,\ldots,Z_n$. Albers, Bickel and Van Zwet (1976, Section 5) discuss these in the case of identically distributed random variables. However, weaker conditions will ensure these results but it is not appropriate to discuss these here. Some discussion appears in John (1981).

5. Confidence Intervals

To find a confidence interval with coefficient $1-\alpha$, we consider the one sided test for the hypothesis concerning particular values of θ_0 or δ_0 and find the significance level of this test. If the significance level is greater than $\alpha/2$, then that value is in the confidence interval. So a confidence interval could be found by obtaining significance levels for a "grid" of values of θ_0 or δ_0. However, a more satisfactory computing procedure is given by the iterative algorithm

$$\theta_{i+1} = \theta_i + [\alpha - s(\theta_i)][(s(\theta_i)-s(\theta_{i-1}))/(\theta_i-\theta_{i-1})]^{-1},$$

where $s(\theta_0)$ is the significance level for a test of the hypothesis $\theta = \theta_0$. Since $s(\theta)$ is monotone for $0<\theta<\max_k|a_k|$, this process converges, and the convergence is rapid if appropriate starting points are taken.

6. Numerical Results

Before considering particular examples, we will consider a continuity correction which leads to some improvement in the approximation for small samples. First, we notice that at $x = \Sigma_k|a_k|$ in the one sample case and for $x = w_N^{-\frac{1}{2}}\Sigma_{k=1}^m c_k$ where $c_1 \geqslant \ldots \geqslant c_{m+n}$, are the ordered values of $a_1,\ldots,a_{m+n}$, in the two sample case, the approximating probabilities are zero, while the true probabilities are 2^{-n} and $\binom{m+n}{m}^{-1}$, respectively. However, at $x = 0$, both the approximations and the true values are $\frac{1}{2}$. Jumps of size 2^{-n} or $\binom{m+n}{m}^{-1}$ occur at various points in the exact distribution but the approximating distribution is continuous. Thus, if x is a point where a jump may occur and P_x is the approximation, then a better approximation is given by

$$P_x + 2^{-n-1} + xM^{-1}2^{-n-1},$$

where $M = \Sigma_k |a_k|$, in the one sample case. A similar result holds in the two sample case.

TABLE 1

Examples of significance levels and confidence intervals for one-sample problems

Data: Difference in sleep gained. Bickel and Doksum (1977, p. 215)
1·2, 2·4, 1·3, 1·3, 0·0, 1·0, 1·8, 0·8, 4·6, 1·4

	Exact	*First saddlepoint*	*Second saddlepoint*	*Edgeworth*
Significance level	0·004	0·002	0·002	0·004
Confidence intervals				
0·989	0·65, 2·95	0·65, 2·98	0·63, 3·04	0·60, 2·87
0·976	0·73, 2·73	0·75, 2·65	0·73, 2·69	0·66, 2·61
0·949	0·85, 2·45	0·86, 2·45	0·84, 2·48	0·76, 2·44

Data: Difference in plant height. Fisher (1935, Section 21)
−67, −48, 6, 8, 14, 16, 23, 24, 28, 29, 41, 49, 56, 60, 75

	Exact	*First saddlepoint*	*Second saddlepoint*	*Edgeworth*
Significance level	0·052	0·048	0·052	0·054
Confidence intervals				
0·990	−9·4, 47·0	−9·0, 46·8	−9·5, 47·2	−8·4, 47·7
0·975	−4·0, 43·6	−3·7, 43·3	−4·2, 43·7	−3·5, 44·3
0·950	−0·2, 41·0	0·2, 40·5	−0·3, 41·1	0·2, 41·4

TABLE 2

Examples of significance levels and confidence intervals for two sample problems

Hours of pain relief due to drugs. Lehmann (1975, p. 37)
A 6·8 3·1 5·8 4·5 3·3 4·7 4·2 4·9
B 4·4 2·5 2·8 2·1 6·6 0·0 4·8 2·3

	Exact	*First saddlepoint*	*Second saddlepoint*	*Edgeworth*
Significance level	0·102	0·089	0·101	0·098
Confidence intervals				
0·991	−1·00, 3·97	−0·96, 3·88	−1·04, 3·95	−0·93, 3·87
0·975	−0·62, 3·53	−0·57, 3·50	−0·64, 3·56	−0·57, 3·51
0·950	−0·30, 3·26	−0·27, 3·22	−0·33, 3·28	−0·29, 3·23

Effect of analgesia for two classes. Lehmann (1975, p. 92)
Class I 17·9 13·3 10·6 7·6 5·7 5·6 5·4 3·3 3·1 0·9
Class II 7·7 5·0 1·7 0·0 −3·0 −3·1 −10·5

	Exact	*First saddlepoint*	*Second saddlepoint*	*Edgeworth*
Significance level	0·012	0·010	0·011	0·014
Confidence intervals				
0·990	−0·10, 16·13	0·06, 15·96	−0·15, 16·19	−0·02, 15·76
0·975	0·92, 14·68	1·18, 14·51	0·98, 14·72	1·07, 14·44
0·950	1·88, 13·52	2·07, 13·46	1·86, 13·64	1·95, 13·45

In Tables 1 and 2, confidence intervals and significance levels are given for data sets which have been used in the literature to illustrate non-parametric methods. These are compared to exact values obtained for the permutation tests and to values obtained using an Edgeworth approximation with three terms. These values have been obtained by John (1981). The first saddlepoint approximation and the three term Edgeworth approximation are of about the same accuracy and the second saddlepoint approximation appears to be slightly better. The error in all these approximations is very close to the order of error caused by the discontinuity of the exact test. The approximations were also compared for one and two sample Wilcoxon tests and similar results were apparent in these cases. All three approximations are of a suitable accuracy for practical application.

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APPENDIX 1

Using (5) and (6) we have

$$B_1 = Q_n(u)\int_{(x_n-m_n)/\sigma_n}^{\infty} \exp(-u\sigma y - um_n)\,\phi(y)\,(1-\tfrac{1}{2}\kappa_{1n}H_1(y)+\tfrac{1}{6}\kappa_{3n}H_3(y))\,dy$$

$$= Q_n(u)\exp(-um_n+\tfrac{1}{2}u^2\sigma_n^2)\int_{u\sigma_n}^{\infty}\phi(z)\,[1-\tfrac{1}{2}\kappa_{1n}(z-u\sigma_n)$$

$$+\tfrac{1}{6}\kappa_{3n}((z-u\sigma_n)^3-3(z-u\sigma_n))]\,dz.$$

Now

$$\int_a^\infty z\phi(z)\,dz = \phi(a),$$

$$\int_a^\infty z^2\,\phi(z)\,dz = a\phi(a)+1-\Phi(a),$$

$$\int_a^\infty z^3\,\phi(z)\,dz = (a^2+2)\,\phi(a).$$

So integrating the above result gives

$$B_1 = Q_n(u)\exp(-um_n+\tfrac{1}{2}u^2\,\sigma_n^2)(1-\Phi(u\sigma_n))\,[1-\tfrac{1}{2}\kappa_{1n}\,W_1(u\sigma_n)+\tfrac{1}{6}\kappa_{3n}\,W_3(u\sigma_n)]$$

where W_1 and W_3 are defined in Section 3.

We will now indicate how this result can be obtained formally from a local limit theorem as advocated by Daniels (1955). Suppose there is a saddlepoint approximation for the density $f_n(x) = F_n'(x)$ given by

$$f_n(x) = \frac{\exp(L_n(u(x))-xu(x))}{[2\pi L_n''(u(x))]^{\frac{1}{2}}}(1+O(n^{-1})),$$

where $L_n(u) = \log Q_n(u)$ and $u(x)$ is defined implicitly as the solution of the equation

$$L_n'(u(x)) = x.$$

Then

$$u'(x)\,L_n''(u(x)) = 1$$

and

$$u''(x)\,L_n''(u(x))+[u'(x)]^2\,L_n'''(u(x)) = 0.$$

If

$$\frac{L_n'''(u)}{[L_n''(u)]^{3/2}} = O(n^{-\frac{1}{2}}),\quad \frac{L_n^{(4)}(u)}{[L_n''(u)]^2} = O(n^{-1}),$$

as in the case when we consider sums of independent, identically distributed, continuous random variables, then we have

$$\begin{aligned} L_n(u(y))-yu(y) &= L_n(u(x_n))-x_n\,u(x_n)-(y-x_n)\,u(x_n) \\ &\quad -\tfrac{1}{2}(y-x_n)^2\,u'(x_n)-\tfrac{1}{6}(y-x_n)^3\,u''(x_n)+O(n^{-1}), \\ [L_n''(u(y))]^{-\frac{1}{2}} &= [L_n''(u(x_n))]^{-\frac{1}{2}}[1-\tfrac{1}{2}\kappa_{3n}(y-x_n)/\sigma_n+O(n^{-1})], \end{aligned}$$

where $\sigma_n^2 = L_n''(u(x_n))$. Using these results in the integral of $f_n(y)$ gives

$$\begin{aligned} 1-F_n(x_n) &= \int_{x_n}^\infty f_n(y)\,dy \\ &= \frac{\exp(L_n(u(x_n))-x_n\,u(x_n))}{\sigma_n(2\pi)^{\frac{1}{2}}}\int_{x_n}^\infty \exp(-(y-x_n)\,u-\tfrac{1}{2}(y-x_n)^2/\sigma_n^2)(1+O(n^{-1})) \\ &\quad \times(1+\tfrac{1}{6}\kappa_{3n}(y-x_n)^3/\sigma_n^3+O(n^{-1})) \\ &\quad \times(1-\tfrac{1}{2}\kappa_{3n}(y-x_n)/\sigma_n+O(n^{-1}))\,dy. \\ &= B_1(1+O(n^{-1})). \end{aligned}$$

APPENDIX 2

It is shown in Robinson (1977) that the complex moment generating function of T_N is

$$Q_N(u+iv) = (2\pi B_{Nm})^{-1} \int_{-\pi}^{\pi} \exp\{\textstyle\sum_k K((u+iv)\, a_k\, w_N^{-\frac{1}{2}} + \alpha + i\theta)\}\, d\theta, \tag{14}$$

where

$$B_{Nm} = \binom{N}{m} p^m (1-p)^{N-m}.$$

If E is the integrand, then

$$\begin{aligned} E = \exp\{&\textstyle\sum_k K_k + i\theta \sum_k K_k' + ivw_N^{-\frac{1}{2}} \sum_k a_k K_k' \\ &-\tfrac{1}{2}\theta^2 \textstyle\sum_k K_k'' - \theta v w_N^{-\frac{1}{2}} \sum_k a_k K_k'' - \tfrac{1}{2} v^2 w_N^{-1} \sum_k a_k^2 K_k''\} \\ &\times [1 + \tfrac{1}{6} i^3 (\theta^3 \textstyle\sum_k K_k''' + 3\theta^2 v w_N^{-\frac{1}{2}} \sum_k a_k K_k''' + 3\theta v^2 w_N^{-1} \sum_k a_k^2 K_k''' \\ &+ v^3 w_N^{-3/2} \textstyle\sum_k a_k^3 K_k''') + R]. \end{aligned}$$

If we choose α to satisfy (10), then the exponent in E is

$$\textstyle\sum_k K_k + ivm_N - \tfrac{1}{2} v^2 \sigma_N^2 - \tfrac{1}{2}(\theta + v w_N^{-\frac{1}{2}} \sum_k a_k K_k'' / \sum_k K_k'')^2 \sum_k K_k''.$$

So integrating in (14) formally, we obtain

$$Q_N(u+iv) = \frac{\exp(\sum_k K_k + ivm_N - \frac{1}{2} v^2 \sigma_N^2)}{(2\pi)^{\frac{1}{2}} B_{Nm} (\sum_k K_k'')^{\frac{1}{2}}} [1 - \tfrac{1}{2} iv\sigma_N \kappa_{1N} + \tfrac{1}{6}(iv)^3 \sigma_N^3 \kappa_{3N} + R'],$$

where κ_{1N} and κ_{3N} are defined by (12) and (13).